Chapter 4

Preliminaries

4.1 Gaussian Measure and Gaussian Processes

Definition 4.1.1 A measure μ on $\mathbf{R}^n$ is Gaussian if there exists a non-negative symmetric $n \times n$ matrix K and a vector $m \in \mathbf{R}^n$ such that

$$\int_{\mathbf{R}^n} e^{i\langle \lambda, x\rangle} \mu(dx) = e^{i\langle \lambda, m\rangle - \frac{1}{2}\langle K\lambda, \lambda\rangle}.$$

The Gaussian measure has a density with respect to the Lebesgue measure if and only if K is non-degenerate in which case the density is

$$\frac{1}{\sqrt{(2\pi)^n \det(K)}} e^{-\frac{1}{2}\langle K^{-1}(x-m),\, x-m\rangle}.$$

The vector m is called its mean, and K is called its covariance operator.

We emphasise that a Gaussian measure is determined by its mean and its covariance oprator. A random variable with a Gaussian distribution is called a Gaussian random variable.

Theorem 4.1.1 *If X is a Gaussian random variable on $\mathbf{R}^d$ with covariance operator K, and $A : \mathbf{R}^d \to \mathbf{R}^n$ a linear map, then AX is a Gaussian random variable with covariance AKA^T.*

Proof We only need to identify $\mathbf{E}\big[e^{i\langle \lambda, AX\rangle}\big]$ for any $\lambda \in \mathbf{R}^n$:

$$\begin{aligned}\mathbf{E}\big[e^{i\langle \lambda, AX\rangle}\big] &= \mathbf{E}\big[e^{i\langle A^T\lambda, X\rangle}\big] \\ &= e^{i\langle A^T\lambda, m\rangle - \frac{1}{2}\langle KA^T\lambda, A^T\lambda\rangle} \\ &= e^{i\langle \lambda, Am\rangle - \frac{1}{2}\langle AKA^T\lambda, \lambda\rangle}.\end{aligned}$$

This shows that X is a Gaussian random variable with mean Am and covariance AKA^T. □

Let Z be a standard Gaussian variable on $\mathbf{R}^d$, $X = (X_1, \dots, X_n)$, K a matrix and C a vector, then $X = KZ + C$ has Gaussian distribution.

One of the nice properties of Gaussian random variables is the following. Let $(X_1, \dots, X_n)$ be Jointly Gaussian random variables. Then they are independent if and only if they are uncorrelated. Linear combinations of jointly Gaussian random variables are jointly Gaussian.

Exercise 4.1.1 If $\{X_1, \dots X_N\}$ are independent random variables with each X_i Gaussian on $\mathbf{R}^d$, and $a_i \in \mathbf{R}$, show that $\sum_{i=1}^N a_i X_i$ is a Gaussian random variable.

There exists a random variable $X = (X_1, X_2)$ with both marginals X_1 and X_2 Gaussian, but X is not Gaussian.

Definition 4.1.2 A function $x : \mathbf{R} \to \mathbf{R}^d$ is locally Hölder continuous of exponent α if

$$\sup_{0 \le |u-v| \le 1} \frac{|x_u - x_v|}{|u-v|^\alpha} < \infty,$$

As is customary for stochastic processes, we denote the variable of x with subscript.

Definition 4.1.3 A stochastic process (X_t) is said to be continuous, if for almost surely all ω, $t \mapsto X_t(\omega)$ is continuous. Similarly, a stochastic process is Hölder continuous if $t \mapsto X_t(\omega)$ is everywhere Hölder continuous, almost surely.

Definition 4.1.4 A continuous stochastic process W_t with initial value x is a Brownian motion is its finite dimensional distributions are given by:

$$\begin{aligned}&\mathbf{P}(x_{t_1} \in A_1, \dots, x_{t_n} \in A_n))\\ &= \int_{A_1} \cdots \int_{A_k} p(t_1, x, y_1) p(t_2 - t_1, y_1, y_2) \dots p(t_k - t_{k-1}, y_{k-1}, y_k) dy_k \dots dy_1.\end{aligned}$$

Here $p(t, x, y) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{|x-y|^2}{2t}}$ is the heat kernel on $\mathbf{R}^d$.

Recall that for any $\alpha < \frac{1}{2}$, a standard Brownian motion has a continuous version which is furthermore locally Hölder continuous of order α.

4.2 A reservoir of stochastic processes