Solution Sheet n°8

1. First observe that the interpretation in $\mathcal{N}$ of each term $t(x_1, \ldots, x_p)$ is a primitive recursive function $h_t : \mathbb{N}^p \to \mathbb{N}$. This is shown by induction. The interpretations of variables and constant are respectively the projection $\mathsf{proj}_1^1 : \mathbb{N} \to \mathbb{N}$ and the constant functions $\mathsf{const}_0^1, \mathsf{const}_1^1 : \mathbb{N} \to \mathbb{N}$ which are by definition primitive recursive. Then for terms t and s whose interpretation are the primitive recursive functions h_t and h_s, the interpretation of the terms $t+s$ and $t \cdot s$ are respectively the primitive recursive functions $\mathsf{add}(h_t, h_s)$ and $\mathsf{mult}(h_t, h_s)$.

   Atomic formulas: Since the usual relations $=$ and $\leq$ on $\mathbb{N}$ are primitive recursive binary relations (as seen during the lecture), for each couple of terms s and t the atomic formulas $t = s$ and $t \leq s$ define the primitive recursive relation whose characteristic functions are $\chi_=(h_s, h_t)$ and $\chi_\leq(h_s, h_t)$, respectively. Thus sets which are arithmetically defined by atomic formulas are primitive recursive.

   Δ_0^0-rudimentary formulas: Suppose now that Δ_0^0-rudimentary formulas φ and ψ (arithemtically) define sets which are primitive recursive. Then the same is true of the formulas $\neg\varphi$, $\varphi \wedge \psi$, $\varphi \vee \psi$ by the fact that primitive recursive sets are closed under complementation, intersection and union. Also, a set which is defined by the formulas $\forall x < t\ \varphi$ or $\exists x < t\ \varphi$ is primitive recursive since it is obtained by bounded quantification

   $$\exists i \leq h_t(\vec{n})\ \ R(\vec{n}, i) \quad \text{or} \quad \forall i \leq h_t(\vec{n})\ \ R(\vec{n}, i)$$

   where h_t is the interpretation of the term t and R is a relation defined by φ. We proved in Sheet 7 that such relations are primitive recursive when both h_t and R are.

2. During the lecture we saw that a set $B \subseteq \mathbb{N}^p$ is recursively enumerable iff there exists $A \subseteq \mathbb{N}^{p+1}$ primitive recursive such that $B = \{\vec{x} \in \mathbb{N}^p \mid \exists y \in \mathbb{N}\ (\vec{x}, y) \in A\}$. Thus by 1. any set which is definable by a formula of the form $\exists x \varphi(x)$ where φ is a Δ_0^0-rudimentary formulas is recursively enumerable.

3. This function is arithmetically defined by the Δ_0^0-rudimentary formula:

   $$\mathsf{quot}(x_1, x_2, y)\ :\ (x_2 = 0 \wedge y = 0) \vee \exists u < x_2 (x_1 = y \cdot x_2 + u).$$

4. This function is arithmetically defined by the Δ_0^0-rudimentary formula:

   $$\mathsf{rest}(x_1, x_2, y)\ :\ (x_2 = 0 \wedge y = x_1) \vee (y < x_2 \wedge \exists u \leq x_1 (x_1 = u \cdot x_2 + y)).$$

5. This follows from 4. and the fact that $(t \cdot (1 + i)) + 1$ is a term of the language $\mathcal{A}$.

6. Let $k \in \mathbb{N}$ and $(n_0, \ldots, n_k) \in \mathbb{N}^{k+1}$. We set $m = \max\{n_0, \ldots, n_k, k\}$ and $t = m!$. We show that for i and j with $0 \leq i < j \leq k$ the natural numbers $t(i+1)+1$ and $t(j+1)+1$ are coprime (i.e. their greatest common divisor

is 1). To this end, suppose that a natural number r divides both $t(i+1)+1$ and $t(j+1)+1$. Then it must divide their difference $t(j-i)$. Thus r divides $j-i$ or $t=m!$. Since $j-i \leq m$, trivially $j-i$ divides $t=m!$, necessarily r divides t. But t and $t(i+1)+1$ are coprime since, if $t=qr$ and $qr(i+1)+1=qr'$ then $q(ri+r-r')=-1$ so $q=\pm 1$. Consequently, r must equal 1 and $t(i+1)+1$ and $t(j+1)+1$ are coprime as desired.

We have thus obtained that the sequence of natural numbers $t+1, 2t+1, \ldots, t(k+1)+1$ is pairwise coprime and thus by the Chinese remainder theorem there exists a natural number s such that for all i with $0 \leq i \leq k$ we have $a_i = \mathsf{rest}(s, t(i+1)+1) = \beta(s,t,i)$.

We can thus use Gödel's β function to code sequences of natural numbers of arbitrary length using just two natural numbers, s and t in the above formulation.

7. Basic recursive function: the constant functions, the projections and the successor function are respectively defined by the Δ_0^0-rudimentary formulas:

$$\begin{aligned} \mathsf{const}_n^p(x_1,\ldots,x_p,y) &: y = n \\ \mathsf{proj}_j^p(x_1,\ldots,x_p,y) &: y = x_j \\ \mathsf{succ}(x,y) &: y = x+1. \end{aligned}$$

Composition: Now suppose that $g : \mathbb{N}^m \to \mathbb{N}$ and $f_1, \ldots, f_m : \mathbb{N}^p \to \mathbb{N}$ are (partial) recursive functions defined by generalised existential Δ_0^0-rudimentary formulas (gen-$\exists$-Δ_0^0-rud) $\varphi_g(x_1,\ldots,x_m,y)$ and $\varphi_{f_i}(x_1,\ldots,x_p,y)$ respectively. Then the partial recursive function $g(f_1,\ldots,f_m)$ is defined by the gen-$\exists$-Δ_0^0-rud formula:

$$\psi(x_1,\ldots,x_p,y) \;:\; \exists y_1 \exists y_2 \cdots \exists y_m \left(\bigwedge_{i=1}^{m} \varphi_{f_i}(x_1,\ldots x_p, y_i) \wedge \varphi_g(y_1,\ldots,y_m,y) \right).$$

Induction: Suppose that $g : \mathbb{N}^p \to \mathbb{N}$ and $h : \mathbb{N}^{p+2} \to \mathbb{N}$ are (partial) recursive functions defined by gen-$\exists$-Δ_0^0-rud formulas $\varphi_g(\vec{x},y)$ and $\varphi_h(\vec{x},y)$ respectively. The function f defined by induction from g and h is defined by the gen-$\exists$-Δ_0^0-rud formula:

$$\begin{aligned} \psi(\vec{x},y,z) \;:\; \exists s \exists t \Big(&\exists y_0 \left(\beta(s,t,0,y_0) \wedge \varphi_g(\vec{x},y_0)\right) \\ &\wedge \\ \forall w < y \exists y_1 \exists y_2 &(\beta(s,t,w,y_1) \wedge \beta(s,t,w+1,y_2) \wedge \varphi_h(\vec{x},w,y_1,y_2)) \\ &\wedge \\ &\beta(s,t,y,z)\Big). \end{aligned}$$

What we have done is find s,t which code the sequence

$$\big(f(\vec{x},0), f(\vec{x},1), \ldots, f(\vec{x},y)\big) = \;= (g(\vec{x}), h(\vec{x},0,g(\vec{x})), \ldots, h(\vec{x},y-1,f(\vec{x},y-1)))$$

and finally check that z is equal to the last element of the sequence, that is $z = f(\vec{x},y)$.

Minimisation: Suppose that $g : \mathbb{N}^{p+1} \to \mathbb{N}$ is a (partial) recursive function defined by a gen-$\exists$-Δ_0^0-rud formula $\varphi_g(\vec{x}, y, z)$. The function $f(\vec{x}) = \mu y g(\vec{x}, y) = 0$ is defined by the gen-$\exists$-Δ_0^0-rud formula

$$\psi(\vec{x}, z) \ : \ \varphi_g(\vec{x}, z, 0) \wedge \forall y < z \ \exists u(\varphi_g(\vec{x}, y, u) \wedge 1 \leq u)).$$

8. (a) A formula φ is logically equivalent to $\exists w(w = w \wedge \varphi)$ for a variable w with no free occurrence in φ. Moreover if φ is Δ_0^0-rud, then so is $w = w \wedge \varphi$.

   (b) $\exists x\varphi(x) \wedge \exists y\psi(y)$ is arithmetically equivalent to

   $$\exists w \exists x < w \ \exists y < w \ (\varphi(x) \wedge \psi(y))$$

   for w with no free occurence in φ and ψ. The backward direction is straightforward, while the forward one follows from the fact that in $\mathcal{N}$ for any two natural numbers n and m there exists a natural number greater than both, that is $\mathcal{N} \models \forall x \forall y \exists w(x < w \wedge y < w)$. Moreover if φ and ψ are Δ_0^0-rud, then so is $\exists x < w \ \exists y < w \ (\varphi(x) \wedge \psi(y))$.

   (c) $\exists x\varphi(x) \vee \exists y\psi(y)$ is logically equivalent to $\exists x(\varphi(x) \vee \psi(x))$.

   (d) $\forall z < t(x_1, \ldots, x_p) \ \exists u\varphi(x_1, \ldots, x_p, z, u)$ is arithmetically equivalent to

   $$\exists w \forall z < t(x_1, \ldots, x_p) \ \exists u < w\varphi(x_1, \ldots, x_p, z, u).$$

   where w has no free occurrence in φ and ψ. The backward direction is straightforward, while the forward one is based on the fact about $\mathcal{N}$ according to which for every finitely many natural numbers $u_0, \ldots, u_{t-1}$ there exists a natural number greater than all these u_z.

   (e) $\exists z < y \ \exists u\varphi(u, z)$ is logically equivalent to $\exists u \exists z < y \ \varphi(u, z)$.

   (f) For similar reasons as in (b), $\exists u \exists v \ \varphi(u, v)$ is arithmetically equivalent to $\exists w \exists u < w \ \exists v < w \ \varphi(u, v)$.

9. First notice that atomic formulas are $\exists\Delta_0^0$-rud by (a) of the previous point. Next recall that gen-$\exists$-Δ_0^0-rud formulas are built up from the atomic formulas by disjunctions, conjunctions, bounded quantifications and existential quantifications. Hence, by (b)-(f) of the previous point one can prove by induction on the height of the formulas that every gen-$\exists$-Δ_0^0-rud is equivalent to an $\exists\Delta_0^0$ formula.

10. By point 1. every set which is definable which by a $\exists\Delta_0^0$-rud formula is recursively enumerable. Conversely, observe that a set is recursively enumerable iff it is the domain of a recursive function. Hence given a recursively enumerable set R there is a recursive function f whose domain is R. By 7. every recursive function is arithmetically definable by a gen-$\exists$-Δ_0^0-rud formula $\varphi_f(\vec{x}, y)$. The domain R of f is thus arithmetically defined by the formula

    $$\varphi_R(\vec{x}) \ : \ \exists y \ \varphi_f(\vec{x}, y)$$

    which is arithmetically equivalent to a $\exists\Delta_0^0$-rud formula by 9. Consequently, every recursively enumerable set is arithmetically definable by $\exists\Delta_0^0$-rud formula.