

Solution Sheet n°6

Solution of exercise 1: Recall from the lecture that the following functions are primitive recursive:

$$\begin{aligned} \text{add} : (n, m) &\mapsto n + m & \text{diff} : (n, m) &\mapsto n \dot{-} m \\ \text{prod} : (n, m) &\mapsto n \cdot m & \text{succ} : n &\mapsto n + 1 \\ n : m &\mapsto n, \text{ for every } n \in \mathbb{N} & \text{proj}_{p,i} : (n_1, \dots, n_p) &\mapsto n_i, 1 \leq i \leq p. \end{aligned}$$

1. For each $n \in \mathbb{N}$, the characteristic function $\chi_{\{n\}}$ of $\{n\}$ is primitive recursive since

$$\chi_{\{n\}} = \text{prod}(\text{diff}(\text{succ}(\text{proj}_{1,1}), n), \text{diff}(\text{succ}(n), \text{proj}_{1,1})),$$

or, with lighter notation, $\chi_{\{n\}}(m) = ((m + 1) \dot{-} n) \cdot ((n + 1) \dot{-} m)$. Also the characteristic function of the empty set is just the constant 0.

Moreover for each p -tuple $\vec{n} = (n_1, \dots, n_p) \in \mathbb{N}^p$, the singleton $\{\vec{n}\}$ is recursive, since its characteristic function can be expressed by

$$\chi_{\{\vec{n}\}} = \text{prod}_p(\chi_{\{n_1\}}(\text{proj}_{p,1}), \dots, \chi_{\{n_p\}}(\text{proj}_{p,p})),$$

where by induction we let $\text{prod}_2 = \text{prod}$ and define $\text{prod}_p : \mathbb{N}^p \rightarrow \mathbb{N}$ by

$$\text{prod}_p = \text{prod}(\text{prod}_p(\text{proj}_{p,1}, \dots, \text{proj}_{p,p-1}), \text{proj}_{p,p}),$$

for $p > 2$. Also the empty subset of $\mathbb{N}^p$ admits the characteristic function $0(\text{proj}_{p,1})$.

Finally we show that if A and B are primitive recursive subsets of $\mathbb{N}^p$, then $A \cup B$ is primitive recursive, which concludes the claim. Indeed we have

$$\chi_{A \cup B} = \text{diff}(1, \text{diff}(1, \text{add}(\chi_A, \chi_B))).$$

2. Let $g : \mathbb{N}^0 \rightarrow \mathbb{N}$ be the constant 1 and $h : \mathbb{N}^2 \rightarrow \mathbb{N}$ be the primitive recursive function $\text{prod}(\text{succ}(\text{proj}_{2,1}), \text{proj}_{2,2})$. Then factorial is obtained by recursion on g and h as

$$\begin{aligned} f(0) &= g = 1 \\ f(y + 1) &= h(y, f(y)) = \text{prod}(\text{succ}(y), f(y)). \end{aligned}$$

3. We have $d(x, y) = (x \dot{-} y) + (y \dot{-} x)$, that is,

$$d = \text{add}(\text{diff}(\text{proj}_{2,1}, \text{proj}_{2,2}), \text{diff}(\text{proj}_{2,2}, \text{proj}_{2,1})).$$

4. By induction on $k \in \mathbb{N}$. Observe that for $k = 0$, $p_0 = \text{proj}_{2,1}$. Now supposing by induction that p_k is primitive recursive we have

$$\begin{aligned} p_{k+1} &= \text{add}(p_k(\text{proj}_{k+3,1}, \dots, \text{proj}_{k+3,k+1}), \text{proj}_{k+3,k+3}), \\ &\quad \text{prod}(\text{proj}_{k+3,k+2}, \text{exp}(\text{proj}_{k+3,k+3}, k + 1)), \end{aligned}$$

where $\text{exp}(m, n) = m^n$ is primitive recursive since it is defined by recursion on the constant 1 and prod as

$$\begin{aligned} \text{exp}(n, 0) &= 1 \\ \text{exp}(n, m + 1) &= \text{prod}(n, \text{exp}(n, m)). \end{aligned}$$

Solution of exercise 2: See for example section 2 of Chapter 5 (on Moodle) in:

R. Cori, D. Lascar, and D.H. Pelletier. *Mathematical logic: a course with exercises. Part 2 : recursion theory, Gödel's theorems, set theory, model theory*. Mathematical Logic: A Course with Exercises. Oxford University Press, 2001.