

Partial Mock Exam (about 75% size) Solutions

An A4 two side sheet of personal notes is allowed.
Points are only given as an indication of the length
and/or the difficulty of each exercise.

Last name:

First name:

Section:

/53 points

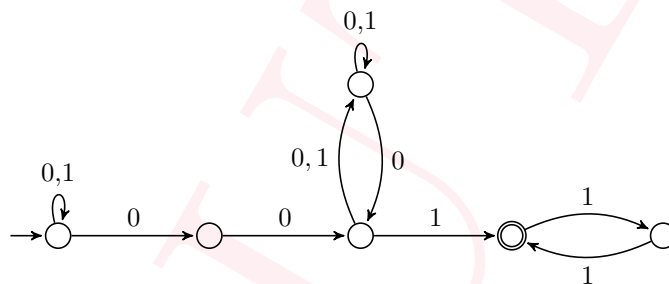
Problem 1: (9 points) Automata, grammars.

2 pt

Question 1.1: Draw the graph of a *nondeterministic* finite automaton which recognises the language

$$L = \{w \in \{0,1\}^* \mid w \text{ contains two consecutive 0 and ends with an odd number of 1}\}.$$

Solution:

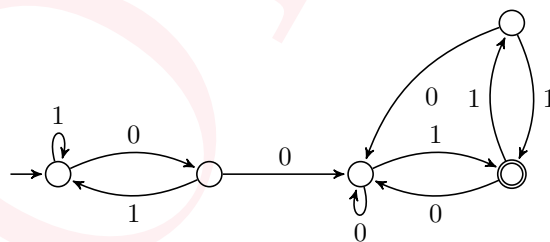


2 pt

Question 1.2: Draw the graph of a *deterministic* finite automaton which recognises the language

$$L = \{w \in \{0,1\}^* \mid w \text{ contains two consecutive 0 and ends with an odd number of 1}\}.$$

Solution:



Let WFF be the language on the alphabet $\{(), \exists, \neg, \vee, =, +, x_1, \dots, x_n\}$ consisting in all the well-formed formulas of first order logic with the one and only $+$ as symbol of binary function and with variables among $x_1, \dots, x_n$.

2 pt

Question 1.3: Show *briefly* that WFF is context free.

Solution:

$$\begin{aligned} S &\rightarrow \exists x_1 S \mid \dots \mid \exists x_n S \mid \neg S \mid (S \vee S) \mid T = T \\ T &\rightarrow (T + T) \mid x_1 \mid \dots \mid x_n \end{aligned}$$

3 pt

Question 1.4: Show *briefly* that WFF is not recognised by a deterministic finite automaton.

Hint: Use the Pumping Lemma¹.

Solution:

Suppose towards contradiction WFF is recognized by a deterministic finite automaton and let p be the natural number which would be given by the pumping lemma. Let

$$w = \underbrace{((\dots (x_1 + x_1) + x_1) + \dots + x_1)}_{p \text{ times}} = x_1).$$

Then for whatever splitting $w = xyz$ satisfying the hypothesis of the pumping lemma, xy^2z would have more left brackets than right ones.

/9 points

¹**Pumping Lemma** Let L be a language on a finite alphabet A recognised by some DFA. There exists a natural number p such that any word $w \in L$ with $|w| \geq p$ can be split into three pieces, $w = xyz$, satisfying the following properties:

1. for all natural number n , $xy^n z \in L$;
2. $|y| > 0$;
3. $|xy| \leq p$.

Problem 2: (16 points) **Recursive sets, decidable sets.**

Read attentively the following:

Definition. Two sets $A \subseteq \mathbb{N}$ and $B \subseteq \mathbb{N}$ are called recursively inseparable if

1. they are disjoint, that is $A \cap B = \emptyset$; and
2. there is no recursive set X such that $A \subseteq X$ and $X \cap B = \emptyset$.

Otherwise A and B are called recursively separable.

2 pt

Question 2.1: Let $A \subseteq \mathbb{N}$ and $B \subseteq \mathbb{N}$ be such that 1. and 2. hold. Show that there exists no recursive set Y such that $Y \cap A = \emptyset$ and $B \subseteq Y$.

Solution:

Otherwise $X = (\mathbb{N} \setminus Y)$ would witness that 2. is false.

2 pt

Question 2.2: Show that any two disjoint recursive (Δ_1^0) sets are recursively separable.

Solution:

Just take one of the two as X .

3 pt

Question 2.3: Let A be recursively enumerable (Σ_1^0). When is it the case that A and its complement $\mathbb{N} \setminus A$ are recursively separable.

Solution:

Exactly when A (and *a fortiori* its complement) is recursive.

3 pt

Question 2.4: Give two disjoint sets $A \in \Sigma_1^0 \setminus \Delta_1^0$ and $B \in \Pi_1^0 \setminus \Delta_1^0$ which are recursively separable.

Hint: You can use without any justification the fact that there exists primitive recursive functions $\alpha_2 : \mathbb{N}^2 \rightarrow \mathbb{N}$ and $\beta_2^1, \beta_2^2 : \mathbb{N} \rightarrow \mathbb{N}$ such that $\alpha_2(\beta_2^1, \beta_2^2) = \text{id}_{\mathbb{N}}$ and $(\beta_2^1(\alpha_2), \beta_2^2(\alpha_2)) = \text{id}_{\mathbb{N}^2}$.

Solution:

Let $S \in \Sigma_1^0 \setminus \Delta_1^0$ and define $A = \alpha_2(\{0\} \times S)$, $B = \alpha_2(\{1\} \times \mathbb{N} \setminus S)$. Then A, B are as required and $X = \alpha_2(\{0\} \times \mathbb{N})$ separates them.

In what follows, we consider a *recursive enumeration* of all Turing machines which compute partial functions from $\mathbb{N}$ to $\mathbb{N}$. According to this enumeration, we call $\mathcal{M}_k$ the k -th Turing machine. For all $k \in \mathbb{N}$, we let φ_k be the partial function from $\mathbb{N}$ to $\mathbb{N}$ computed by $\mathcal{M}_k$.

Consider the two following sets of natural numbers

$$A = \{k \in \mathbb{N} \mid \varphi_k(k) = 0\} \quad \text{and} \quad B = \{k \in \mathbb{N} \mid \varphi_k(k) = 1\}.$$

3 pt

Question 2.5: Explain briefly why A (and B) are Turing recognisable.

Solution:

Let $\mathcal{M}$ simulate φ_k on input k and accept if the simulated computation halts and prints 0. Then $\mathcal{M}$ recognizes A . Similarly for B .

3 pt

Question 2.6: Show that A and B are recursively inseparable.

Hint: Towards a contradiction, suppose that a recursive set X separates A and B . And then, consider the characteristic function of X .

Solution:

Towards contradiction let X separate A, B and let k be such that φ_k is the characteristic function of X . If $k \in X$ then $\varphi_k(k) = 1$, so $k \in B$. If $k \notin X$ then $\varphi_k(k) = 0$, so $k \in A$. A contradiction.

/16 points

Problem 3: (12 points) **Incompleteness of $\mathcal{Rob}$.**

We recall that any recursively enumerable set $A \subseteq \mathbb{N}^k$ is representable by a Σ_1 formula in $\mathcal{Rob}$, namely there is an arithmetic formula $\varphi(x_0, \dots, x_{k-1})$ with free variable among $x_0, \dots, x_{k-1}$ such that for every $(n_0, \dots, n_{k-1}) \in \mathbb{N}^k$:

- if $(n_0, \dots, n_{k-1}) \in A$, then $\mathcal{Rob} \vdash \varphi(n_0, \dots, n_{k-1})$,
- if $(n_0, \dots, n_{k-1}) \notin A$, then $\mathcal{Rob} \vdash \neg\varphi(n_0, \dots, n_{k-1})$.

4 pt

Question 3.1: Let $\varphi(x_0)$ be an arithmetic formula with free variable x_0 . Show that the set

$$X = \{n \in \mathbb{N} \mid \mathcal{Rob} \vdash \varphi(n)\}$$

is recursively enumerable.

Solution:

Using the recursive set

$$\text{Proof}_{\mathcal{Rob}} = \left\{ (m, n) \in \mathbb{N}^2 \mid \begin{array}{l} n \text{ is the code of a proof from } \mathcal{Rob} \\ \text{of the closed formula coded by } m. \end{array} \right\}.$$

we obtain

$$X = \{m \in \mathbb{N} \mid \exists n \in \mathbb{N} (\varphi(m), n) \in \text{Proof}_{\mathcal{Rob}}\}.$$

Now since $m \mapsto \varphi(m) = \text{subs}(0, n, \varphi)$ is a recursive function, it follows that X is recursively enumerable.

4 pt

Question 3.2: Let $A \subseteq \mathbb{N}^2$ be recursive, and let

$$B = \{n \in \mathbb{N} \mid \exists m \in \mathbb{N} (n, m) \in A\}.$$

Show that there exists an arithmetic formula $\psi(x)$ with x as only free variable such that for all $n \in \mathbb{N}$

$$n \in B \iff \mathcal{Rob} \vdash \psi(n).$$

Solution:

Let $\varphi(x_0, x_1)$ represent A in $\mathcal{Rob}$. Let $n \in B$, so that for some $m \in \mathbb{N}$ $(n, m) \in A$. We have $\mathcal{Rob} \vdash \varphi(n, m)$. Therefore in any model $\mathcal{M}$ of $\mathcal{Rob}$, we have $\mathcal{M} \models \varphi(n, m)$ and therefore $\mathcal{M} \models \exists x \varphi(n, x)$. Conversely if $n \notin B$ then $\mathbb{N} \models \neg \exists x \varphi(n, x)$, and therefore $\mathcal{Rob} \not\vdash \exists x \varphi(n, x)$. The formula $\psi(y) : \exists x \varphi(y, x)$ is as desired.

4 pt

Question 3.3: Let $A \subseteq \mathbb{N}^2$ be recursive such that

$$B = \{n \in \mathbb{N} \mid \exists m \in \mathbb{N} (n, m) \in A\}$$

is not recursive and let $\psi(x)$ be given for B by the previous question. Show that there exists $m \in \mathbb{N}$ such that neither $\mathcal{R}ob \vdash \psi(m)$ nor $\mathcal{R}ob \vdash \neg\psi(m)$.

Solution:

We have $B = \{n \in \mathbb{N} \mid \mathcal{R}ob \vdash \psi(n)\}$ is recursively enumerable not recursive. We let $C = \{n \in \mathbb{N} \mid \mathcal{R}ob \vdash \neg\psi(n)\}$. Since $\mathcal{R}ob$ is assumed consistent, B and C are disjoint. Moreover C is not the complement of B in $\mathbb{N}$. Indeed, by Question 1 the set C is recursively enumerable, so if C was the complement of B , which is recursively enumerable, then B would be recursive, but this is not the case. So we can pick some $m \in \mathbb{N}$ with $m \notin B$ and $m \notin C$. If $\mathcal{R}ob \vdash \psi(m)$, then $m \in B$. If $\mathcal{R}ob \vdash \neg\psi(m)$, then $m \in C$. Hence $\psi(m)$ is undecidable in $\mathcal{R}ob$.

/12 points

Problem 4: (16 points) **Undecidability.**

We consider the Gödel numbering φ of arithmetic formulas φ as presented during the lecture. We let $\phi_{proof_{\mathcal{R}ob}}(x, y)$ be a Σ_1^0 arithmetic formula which represents in $\mathcal{R}ob$ the primitive recursive set of pairs of natural numbers

$$\text{Proof}_{\mathcal{R}ob} = \{(P, \varphi) \in \mathbb{N}^2 \mid P \text{ is a proof of } \mathcal{R}ob \vdash_c \varphi\}.$$

Therefore by definition for all $(p, n) \in \mathbb{N}^2$

$$\begin{aligned} (p, n) \in \text{Proof}_{\mathcal{R}ob} &\text{ implies } \mathcal{R}ob \vdash \phi_{proof_{\mathcal{R}ob}}(p, n), \text{ and} \\ (p, n) \notin \text{Proof}_{\mathcal{R}ob} &\text{ implies } \mathcal{R}ob \vdash \neg \phi_{proof_{\mathcal{R}ob}}(p, n). \end{aligned}$$

We also assume we have a coding of 1-tape Turing machines with input alphabet $\{0, 1\}$ into natural numbers. We write $\ulcorner \mathcal{M} \urcorner$ for the code of $\mathcal{M}$. Our coding is assumed to have the property that there is a 1-tape Turing machine $\mathcal{F}$ which when given the binary code of $\ulcorner \mathcal{M} \urcorner$ as input computes the (binary) code of a Δ_0^0 arithmetic formula $C_{\mathcal{M}}(x_0, x_1)$ such that for all $n \in \mathbb{N}$

$$\mathcal{M} \text{ accepts the binary code of } n \text{ iff } \mathbb{N} \models \exists x_0 C_{\mathcal{M}}(x_0, n).$$

Let

$$\begin{aligned} T_0 &= \{n \mid n \text{ is the code of a theorem of } \mathcal{R}ob\}, \\ T_0^1 &= \{n \mid n \text{ is the code of a } \Sigma_1 \text{ theorem of } \mathcal{R}ob\}, \\ \mathcal{H} &= \{n \mid n \text{ is the code of a TM which halts on } 0\}. \end{aligned}$$

Recall that $\mathcal{R}ob$ is Σ_1 -complete, that is, if φ is a Σ_1 arithmetic sentence which is true in $\mathbb{N}$, then $\mathcal{R}ob \vdash \varphi$.

5 pt

Question 4.1: Show that T_0^1 is decidable iff T_0 is decidable.

Solution:

Suppose T_0 is decidable. Then to decide whether $n \in T_0^1$ it is enough to check whether $n \in T_0$ and n is the code of a well formed closed Σ_1 formula. Conversely, if T_0^1 is decidable, $n \in T_0$ if and only if $\mathbb{N} \models \exists p \phi_{proof_{\mathcal{R}ob}}(p, n)$ if and only if $\mathcal{R}ob \vdash \exists p \phi_{proof_{\mathcal{R}ob}}(p, n)$ if and only if $\exists p \phi_{proof_{\mathcal{R}ob}}(p, n) \in T_0^1$

5 pt

Question 4.2: Show that if T_0^1 is decidable then $\mathcal{H}$ is decidable.

Solution:

$n \in \mathcal{H} \iff \mathbb{N} \models \exists x_0 C_{\mathcal{M}}(x_0, n) \iff \mathcal{R}ob \vdash \exists x_0 C_{\mathcal{M}}(x_0, n) \iff \exists x_0 C_{\mathcal{M}}(x_0, n) \in T_0^1$. Given n , compute $\exists x_0 C_{\mathcal{M}}(x_0, n)$ via $\mathcal{F}$ (plus a bit of computation).

5 pt

Question 4.3: Show that if $\mathcal{H}$ is decidable then T_0 is decidable.

Solution:

Since T_0 is recursively enumerable there exists a TM $\mathcal{M}$ such that $n \in T_0$ if and only if $\mathcal{M}(n)$ halts and accepts. Given n , compute the code for the TM $\mathcal{M}'_n$ which on input 0 writes n on the tape and simulates $\mathcal{M}$ (but loops if $\mathcal{M}$ rejects). Then $n \in T_0$ if and only if $\ulcorner \mathcal{M}'_n \urcorner \in \mathcal{H}$.

1 pt

Question 4.4: Conclude that T_0 , T_0^1 , $\mathcal{H}$ are all undecidable.

Solution:

$\mathcal{H}$ is, by Turing's halting problem.

/16 points