Partial Mock Exam (about 75% size)

An A4 two side sheet of personal notes is allowed.
Points are only given as an indication of the length and/or the difficulty of each exercise.

Last name:

First name:

Section:

/53 points

Problem 1: (9 points) **Automata, grammars.**

2 pt **Question 1.1:** Draw the graph of a *nondeterministic* finite automaton which recognises the language

$L = \{w \in \{0,1\}^* \mid w$ contains two consecutive 0 and ends with an odd number of 1$\}$.

2 pt **Question 1.2:** Draw the graph of a *deterministic* finite automaton which recognises the language

$L = \{w \in \{0,1\}^* \mid w$ contains two consecutive 0 and ends with an odd number of 1$\}$.

Let WFF be the language on the alphabet $\{), (, \exists, \neg, \vee, =, +, x_1, \ldots, x_n\}$ consisting in all the well-formed formulas of first order logic with the one and only $+$ as symbol of binary function and with variables among $x_1, \ldots, x_n$.

2 pt **Question 1.3:** Show *briefly* that WFF is context free.

3 pt **Question 1.4:** Show *briefly* that WFF is not recognised by a deterministic finite automaton.

Hint: Use the Pumping Lemma[1].

/9 points

[1]**Pumping Lemma** *Let L be a language on a finite alphabet A recognised by some DFA. There exists a natural number p such that any word $w \in L$ with $|w| \geqslant p$ can be split into three pieces, $w = xyz$, satisfying the following properties:*

1. *for all natural number n, $xy^n z \in L$;*
2. $|y| > 0$*;*
3. $|xy| \leqslant p$*.*

<u>Problem 2</u>: (16 points) **Recursive sets, decidable sets.**

Read attentively the following:

Definition. *Two sets* $A \subseteq \mathbb{N}$ *and* $B \subseteq \mathbb{N}$ *are called* recursively inseparable *if*

1. *they are disjoint, that is* $A \cap B = \emptyset$*; and*
2. *there is no recursive set* X *such that* $A \subseteq X$ *and* $X \cap B = \emptyset$*.*

Otherwise A *and* B *are called* recursively separable*.*

2 pt **Question 2.1:** Let $A \subseteq \mathbb{N}$ and $B \subseteq \mathbb{N}$ be such that 1. and 2. hold. Show that there exists no recursive set Y such that $Y \cap A = \emptyset$ and $B \subseteq Y$.

2 pt **Question 2.2:** Show that any two disjoint recursive (Δ^0_1) sets are recursively separable.

3 pt **Question 2.3:** Let A be recursively enumerable (Σ^0_1). When is it the case that A and its complement $\mathbb{N} \backslash A$ are recursively separable.

3 pt **Question 2.4:** Give two disjoint sets $A \in \Sigma^0_1 \backslash \Delta^0_1$ and $B \in \Pi^0_1 \backslash \Delta^0_1$ which are recursively separable.

Hint: You can use without any justification the fact that there exists primitive recursive functions $\alpha_2 : \mathbb{N}^2 \to \mathbb{N}$ and $\beta^1_2, \beta^2_2 : \mathbb{N} \to \mathbb{N}$ such that $\alpha_2(\beta^1_2, \beta^2_2) = \mathrm{id}_{\mathbb{N}}$ and $(\beta^1_2(\alpha_2), \beta^2_2(\alpha_2)) = \mathrm{id}_{\mathbb{N}^2}$.

In what follows, we consider a *recursive enumeration* of all Turing machines which compute partial functions from $\mathbb{N}$ to $\mathbb{N}$. According to this enumeration, we call $\mathcal{M}_k$ the k-th Turing machine. For all $k \in \mathbb{N}$, we let φ_k be the partial function from $\mathbb{N}$ to $\mathbb{N}$ computed by $\mathcal{M}_k$.

Consider the two following sets of natural numbers

$$A = \{k \in \mathbb{N} \mid \varphi_k(k) = 0\} \quad \text{and} \quad B = \{k \in \mathbb{N} \mid \varphi_k(k) = 1\}.$$

3 pt **Question 2.5:** Explain briefly why A (and B) are Turing recognisable.

3 pt **Question 2.6:** Show that A and B are recursively inseparable.

Hint: Towards a contradiction, suppose that a recursive set X separates A and B. And then, consider the characteristic function of X.

/16 points

Problem 3: (12 points) Incompleteness of $\mathcal{R}ob$.

We recall that any recursively enumerable set $A \subseteq \mathbb{N}^k$ is representable by a Σ_1 formula in $\mathcal{R}ob$, namely there is an arithmetic formula $\varphi(x_0, \ldots, x_{k-1})$ with free variable among $x_0, \ldots, x_{k-1}$ such that for every $(n_0, \ldots, n_{k-1}) \in \mathbb{N}^k$:

- if $(n_0, \ldots, n_{k-1}) \in A$, then $\mathcal{R}ob \vdash \varphi(n_0, \ldots, n_{k-1})$,
- if $(n_0, \ldots, n_{k-1}) \notin A$, then $\mathcal{R}ob \vdash \neg\varphi(n_0, \ldots, n_{k-1})$.

4 pt **Question 3.1:** Let $\varphi(x_0)$ be an arithmetic formula with free variable x_0. Show that the set

$$X = \{n \in \mathbb{N} \mid \mathcal{R}ob \vdash \varphi(n)\}$$

is recursively enumerable.

4 pt **Question 3.2:** Let $A \subseteq \mathbb{N}^2$ be recursive, and let

$$B = \{n \in \mathbb{N} \mid \exists m \in \mathbb{N}\ (n, m) \in A\}.$$

Show that there exists an arithmetic formula $\psi(x)$ with x as only free variable such that for all $n \in \mathbb{N}$

$$n \in B \quad \longleftrightarrow \quad \mathcal{R}ob \vdash \psi(n).$$

4 pt **Question 3.3:** Let $A \subseteq \mathbb{N}^2$ be recursive such that

$$B = \{n \in \mathbb{N} \mid \exists m \in \mathbb{N}\ (n, m) \in A\}$$

is not recursive and let $\psi(x)$ be given for B by the previous question. Show that there exists $m \in \mathbb{N}$ such that neither $\mathcal{R}ob \vdash \psi(m)$ nor $\mathcal{R}ob \vdash \neg\psi(m)$.

/12 points

Problem 4: (16 points) **Undecidability.**

We consider the Gödel numbering φ of arithmetic formulas φ as presented during the lecture. We let $\phi_{proof_{\mathcal{R}ob}}(x, y)$ be a Σ_1^0 arithmetic formula which represents in $\mathcal{R}ob$ the primitive recursive set of pairs of natural numbers

$$\text{Proof}_{\mathcal{R}ob} = \left\{(P, \varphi) \in \mathbb{N}^2 \,\middle|\, P \text{ is a proof of } \mathcal{R}ob \vdash_c \varphi\right\}.$$

Therefore by definition for all $(p, n) \in \mathbb{N}^2$

$$\begin{aligned} &(p, n) \in \text{Proof}_{\mathcal{R}ob} \text{ implies } \mathcal{R}ob \vdash \phi_{proof_{\mathcal{R}ob}}(p, n), \text{ and} \\ &(p, n) \notin \text{Proof}_{\mathcal{R}ob} \text{ implies } \mathcal{R}ob \vdash \neg\phi_{proof_{\mathcal{R}ob}}(p, n). \end{aligned}$$

We also assume we have a coding of 1-tape Turing machines with input alphabet $\{0, 1\}$ into natural numbers. We write $\ulcorner\mathcal{M}\urcorner$ for the code of $\mathcal{M}$. Our coding is assumed to have the property that there is a 1-tape Turing machine $\mathcal{F}$ which when given the binary code of $\ulcorner\mathcal{M}\urcorner$ as input computes the (binary) code of a Δ_0^0 arithmetic formula $C_{\mathcal{M}}(x_0, x_1)$ such that for all $n \in \mathbb{N}$

$$\mathcal{M} \text{ accepts the binary code of } n \quad \text{iff} \quad \mathbb{N} \models \exists x_0 C_{\mathcal{M}}(x_0, n).$$

Let

$$\begin{aligned} \mathrm{T}_0 &= \{n \mid n \text{ is the code of a theorem of } \mathcal{R}ob\}, \\ \mathrm{T}_0^1 &= \{n \mid n \text{ is the code of a } \Sigma_1 \text{ theorem of } \mathcal{R}ob\}, \\ \mathcal{H} &= \{n \mid n \text{ is the code of a TM which halts on } 0\}. \end{aligned}$$

Recall that $\mathcal{R}ob$ is Σ_1-complete, that is, if φ is a Σ_1 arithmetic sentence which is true in $\mathbb{N}$, then $\mathcal{R}ob \vdash \varphi$.

5 pt **Question 4.1:** Show that T_0^1 is decidable iff T_0 is decidable.

5 pt **Question 4.2:** Show that if T_0^1 is decidable then $\mathcal{H}$ is decidable.

5 pt **Question 4.3:** Show that if $\mathcal{H}$ is decidable then T_0 is decidable.

1 pt **Question 4.4:** Conclude that T_0, T_0^1, $\mathcal{H}$ are all undecidable.

/16 points