Introduction

The basic requirements for this course are contained in the "Mathematical Logic" course. Among other things, you should have a clear understanding of each of the following: first order language, signature, terms, formulas, theory, proof theory, models, completeness theorem, compactness theorem, Löwenheim-Skolem theorem.
It makes no sense to take this course without this solid background on first order logic.

The title of the course is "Gödel and Recursivity" but it should rather be "Recursivity and Gödel" since that is the way we are going to go through these topics

(1) Recursivity

(2) Gödel's incompleteness theorems (there are two of them)

Recursivity is at the heart of computer science, it represents the mathematical side of what computing is like. It is related to arithmetics and to proof theory.
Gödel's incompleteness theorems are concerned with number theory (arithmetics) which itself lies at the core of mathematics. They contradict the commonly shared idea that everything that is true can be proved. It ruins the plan, for every mathematical statement φ to either prove it or disprove it (by proving $\neg\varphi$).
Gödel's 1^{st} incompleteness theorem says that there exists a formula φ from number theory such that neither φ nor $\neg\varphi$ is provable. More precisely, it says that in Peano Arithmetics (which is a first order axiomatization of arithmetics) there exists a formula φ that cannot be proved nor disproved, and if we were to add this formula to Peano Arithmetics, one would find a second one that would not be provable nor disprovable inside their first extension of Peano Arithmetics. And if this new formula would be added again, we could find a third one and so on and so forth. To put it differently, if we want to extend Peano Arithmetics to a larger theory which is *complete* in the sense that it proves or disproves any given formula, there would not have any understanding of this theory, we would not get hold of it, for it would not be *recursive*, meaning that we would not have any efficient way of figuring out whether a given closed formula is part of the theory or not (not provable from the theory, but simply part of the theory!). This is precisely where the notion of recursivity plays a crucial role. One does not have the right comprehension of Gödel's incompleteness theorems without a proper understanding of what recursivity is like.
The formula that we will construct (the one that is not provable nor disprovable in Peano Arithmetics) is rather odd. There is no chance that one might tumble over such a formula during the usual mathematical practice.
However, since Gödel's incompleteness theorem was proved, there have been several examples of real arithmetic mathematical formulas that are not provable nor disprovable in Peano Arithmetics, although the are formulated in the language of arithmetics.
A good example of such a formula is the one related to *Goodstein sequences* (1944). A Goodstein sequence is of the form $G^m_{(0)}, G^m_{(1)}, G^m_{(2)}, \ldots$, etc., where m is a positive integer. It is defined the following way (we take $m = 4$ as an example, the general case being obtained by replacing $G^4_{(0)} = 4$ by $G^m_{(0)} = m$ and gathering the other values $G^m_{(1)}, G^m_{(2)}$, etc., the same way):

- $G^4_{(0)} = 4$. Then, to get $G^4_{(1)}$, write 4 in hereditary base 2: $4 = 2^2$, replace all 2's by 3's, then subtract 1: $3^3 - 1 = 27 - 1 = 26$
- $G^4_{(1)} = 26$. Then, to get $G^4_{(2)}$ write 26 in hereditary base 3: $26 = 2 \cdot 3^2 + 2 \cdot 3^1 + 2 \cdot 3^0$ replace all 3's by 4's, then subtract 1: $2 \cdot 4^2 + 2 \cdot 4^1 + 2 \cdot 4^0 - 1 = 41$
- $G^4_{(2)} = 41$. Then, to get $G^4_{(3t)}$, write 41 in hereditary base 4: $41 = 2 \cdot 4^2 + 2 \cdot 4^1 + 2 \cdot 4^0$ replace all 4's by 5's, then subtract 1: $2 \cdot 5^2 + 2 \cdot 5^1 + 2 \cdot 5^0 - 1 = 60$
- $G^4_{(3)} = 60 \ldots$ etc

Amazingly, $G^4_{(n)}$ increases until n reaches the value $3 \cdot 2^{402653209}$ where it reaches the maximum of $3 \cdot 2^{402653210} - 1$, it stays there for the next $3 \cdot 2^{402653209}$ steps then starts its final descent and eventually reaches 0.
Amazingly, for every integer m, the Goodstein sequence $(G^m_{(n)})_{n \in \mathbb{N}}$ is ultimately constant with value 0, i.e.,

$$\lim_{n \to \infty} G^m_{(n)} = 0.$$

However this statement which is easily formalizable in the language of arithmetics is *not provable* in Peano Arithmetics (Kirby and Paris 1982). It requires a stronger theory to be proved (for instance *second order arithmetic*).
Gödel's 2nd incompleteness theorem then, says that mathematics cannot prove its own consistency (unless it is inconsistent in which case it can prove its own consistency for it can prove everything). More precisely, in any *recursive* extension $\mathcal{T}$ of Peano Arithmetics, the formula "$Cons(\mathcal{T})$" (which is a formula from number theory that asserts that there is no proof of $\perp$ from $\mathcal{T}$) is not provable unless $\mathcal{T}$ is inconsistent i.e., using the formalism of sequent calculus:

$$\textit{If } \mathcal{T} \nvdash_c \, , \textit{ then } \mathcal{T} \nvdash_c Cons(\mathcal{T}).$$

We will present two different approaches:

- Computer Science $\rightsquigarrow$ Turing Machine (one of the abstract model of computer);
- Arithmetic $\rightsquigarrow$ Recursive functions — which are particular functions: $\mathbb{N}^k \to \mathbb{N}$.