Exercise Sheet n°8

Let $\mathcal{A}$ be the first order language of arithmetic with non logical symbols

$$0, 1, +, \cdot, \leq .$$

Its standard interpretation is the structure $\mathcal{N} = \langle \mathbb{N}, 0, 1, +, \cdot, \leq \rangle$ of the natural numbers with the usual arithmetical operations and linear order.

Definition. *A formula $\varphi(x_1, \ldots, x_p)$ of $\mathcal{A}$* arithmetically defines *a set $S \subseteq \mathbb{N}^p$ of p-tuples of natural numbers if for all $n_1, \ldots, n_p \in \mathbb{N}$,*

$$(n_1, \ldots, n_p) \in S \quad \textit{iff} \quad \mathcal{N} \models \varphi(\underline{n_1}, \ldots, \underline{n_p})$$

where $\underline{n}$ stands for $\underbrace{1 + 1 + \cdots + 1}_{n \text{ times}}$.

A formula $\varphi(x_1, \ldots, x_p, y)$ of $\mathcal{A}$ arithmetically defines *a function $f : \mathbb{N}^p \to \mathbb{N}$ if for all $n_1, \ldots, n_p, m \in \mathbb{N}$,*

$$f(n_1, \ldots, n_p) = m \quad \textit{iff} \quad \mathcal{N} \models \varphi(\underline{n_1}, \ldots, \underline{n_p}, \underline{m}),$$

that is φ arithmetically defines the graph of f.

A function or a set is said to be arithmetically definable *if there exists a formula of $\mathcal{A}$ that defines it.*

We define the set of Δ^0_0*-rudimentary formulas* as the set of formulas on the language $\mathcal{A}$ which are built up from atomic formulas using only negation, conjunction, disjunction and bounded quantifications $\forall x < t$ and $\exists x < t$, where t is any term of the language not containing the variable x.

1. Show that for all $p \in \mathbb{N}$ and for any set $S \subseteq \mathbb{N}^p$, if S is arithmetically defined by a Δ^0_0-rudimentary formula, then S is primitive recursive.

We say that a formula is $\exists\Delta^0_0$-rud if it is of the form $\exists x \varphi(x)$ with φ a Δ^0_0rudimentary formula. The aim of the following points is to show that the recursively enumerable sets are exactly the sets which are arithmetically definable by $\exists\Delta^0_0$-rud formulas. We can already make one easy but important step:

2. Show that the sets which are defined by $\exists\Delta^0_0$-rud formulas are recursively enumerable.

These points are of importance further on.

3. Show that the function $\mathsf{quot}(n, k) = \begin{cases} \text{quotient of } n \text{ by } k & \text{if } k \neq 0; \\ 0 & \text{otherwise;} \end{cases}$ is definable by a Δ^0_0-rudimentary formula.

4. Show that the function $\mathsf{rest}(n, k) = \begin{cases} \text{rest of } n \text{ by } k & \text{if } k \neq 0; \\ n & \text{otherwise;} \end{cases}$ is definable by a Δ^0_0-rudimentary formula.

We define Gödel's β function by $\beta(s,t,i) = \mathsf{rest}(s, t(1+i)+1)$.

5. Show that the β function is definable by a Δ_0^0-rudimentary formula.

We recall the following classical result of algebra:

Theorem (Chinese remainder theorem)**.** *Suppose $n_0, n_1, \ldots, n_k$ are positive integers which are pairwise coprime. Then, for any given sequence of integers $a_0, a_1, \ldots, a_k$ there exists an integer x solving the system of simultaneous congruences*

$$x \equiv a_0 \pmod{n_0}, \quad x \equiv a_1 \pmod{n_1}, \quad \ldots, \quad x \equiv a_k \pmod{n_k}.$$

6. Show that for every k and every $a_0, \ldots, a_k$ there exist s and t such that for all i with $0 \le i \le k$ we have $\beta(s,t,i) = a_i$.

We call a formula φ of $\mathcal{A}$ a *generalised existential Δ_0^0-rudimentary formula* if φ is built up from atomic formulas using only conjunction, disjunction, bounded quantification and unbounded existential quantification.

7. Show that any recursive function is definable by generalised existential Δ_0^0-rudimentary formula.

Two arithmetical formulas $\varphi(x_1, \ldots, x_k)$ and $\psi(x_1, \ldots, x_k)$ are said to be *arithmetically equivalent* if for all $n_1, \ldots, n_k \in \mathbb{N}$,

$$\mathcal{N} \models \varphi(n_1, \ldots, n_k) \quad \text{iff} \quad \mathcal{N} \models \psi(n_1, \ldots, n_k)$$

or equivalently,

$$\mathcal{N} \models \forall x_1 \forall x_2 \cdots \forall x_k (\varphi(x_1, \ldots, x_k) \leftrightarrow \psi(x_1, \ldots, x_k)).$$

8. Show the following closure properties of $\exists\Delta_0^0$-rud formulas:
   (a) Any Δ_0^0-rudimentary formula is arithmetically equivalent to a $\exists\Delta_0^0$-rud formula;
   (b) The conjunction of two $\exists\Delta_0^0$-rud formulas is arithmetically equivalent to an $\exists\Delta_0^0$-rud formula;
   (c) The disjunction of two $\exists\Delta_0^0$-rud formulas is arithmetically equivalent to an $\exists\Delta_0^0$-rud formula;
   (d) The result of applying bounded universal quantification to an $\exists\Delta_0^0$-rud formula is arithmetically equivalent to a $\exists\Delta_0^0$-rud formula;
   (e) The result of applying bounded existential quantification to an $\exists\Delta_0^0$-rud formula is arithmetically equivalent to a $\exists\Delta_0^0$-rud formula;
   (f) The result of applying existential quantification to an $\exists\Delta_0^0$-rud formula is arithmetically equivalent to a $\exists\Delta_0^0$-rud formula.

9. Conclude that every generalised existential Δ_0^0-rudimentary formula is arithmetically equivalent to an $\exists\Delta_0^0$ formula.

10. Conclude that the recursively enumerable sets are exactly the sets defined by $\exists\Delta_0^0$ formulas.

    Hint: You can make use of the fact that a set is recursively enumerable if and only if it is the domain of a recursive function.