

Exercise Sheet n°6

Exercise 1:

Show that the following sets or functions are primitive recursive.

1. Any finite subset of $\mathbb{N}^p$, for $p \in \mathbb{N}$.
2. The factorial function $n \mapsto n!$.
3. The distance function $d : \mathbb{N}^2 \rightarrow \mathbb{N}$, $(m, n) \mapsto |m - n|$.
4. For any $k \in \mathbb{N}$, the function $p_k : \mathbb{N}^{k+1} \times \mathbb{N} \rightarrow \mathbb{N}$ defined by $p_k(a_0, \dots, a_k, n) = \sum_{i=0}^k a_i n^i$.

Exercise 2:

The aim of this exercise is to show step by step that a certain total function is not primitive recursive while it is Turing computable. This is a function $\xi : \mathbb{N}^2 \rightarrow \mathbb{N}$ called the Ackermann function and defined by the following clauses

- (i) $\forall x \in \mathbb{N}, \xi(0, x) = 2^x$;
- (ii) $\forall y \in \mathbb{N}, \xi(y, 0) = 1$;
- (iii) $\forall x, y \in \mathbb{N}, \xi(y + 1, x + 1) = \xi(y, \xi(y + 1, x))$.

For each $n \in \mathbb{N}$, we define the function $\xi_n : \mathbb{N} \rightarrow \mathbb{N}$ by $\xi_n(x) = \xi(n, x)$.

1. Show the following claims.
 - (a) For all $n \geq 1$, the function ξ_n is defined by recursion on the constant 1 and the function ξ_{n-1} ;
 - (b) For all $n \in \mathbb{N}$, the function ξ_n is primitive recursive;
 - (c) For all $n \in \mathbb{N}$ and all $x \in \mathbb{N}$, $\xi_n(x) > x$;
Hint: Proceed by induction on n and x simultaneously.
 - (d) For all $n \in \mathbb{N}$, the function ξ_n is strictly increasing;
 - (e) For all $n \geq 1$ and all $x \in \mathbb{N}$, $\xi_n(x) \geq \xi_{n-1}(x)$;
 - (f) Let ξ_n^k be the k^{th} iteration of ξ_n , that is $\xi_n^0 = \text{id}_{\mathbb{N}}$ and for all $k \in \mathbb{N}$, $\xi_n^{k+1} = \xi_n(\xi_n^k)$. For all m, n, k and x , $\xi_n^k(x) < \xi_n^{k+1}(x)$, $\xi_n^k(x) \geq x$, and if $m \leq n$, $\xi_m^k(x) \leq \xi_n^k(x)$.

2. For functions $g : \mathbb{N}^p \rightarrow \mathbb{N}$ and $f : \mathbb{N} \rightarrow \mathbb{N}$ we say that f *dominates* g if there exists a natural number B such that for all $(n_1, \dots, n_p) \in \mathbb{N}^p$

$$g(n_1, \dots, n_p) \leq f(\sup(n_1, \dots, n_p, B)).$$

For each $n \in \mathbb{N}$, we let C_n be the set of all functions from $\mathbb{N}^p$, for some $p \in \mathbb{N}$, into $\mathbb{N}$ which are dominated by an iteration of ξ_n . Formally:

$$C_n = \{f : \mathbb{N}^p \rightarrow \mathbb{N} \mid p \in \mathbb{N} \text{ and } \exists k \in \mathbb{N} (\xi_n^k \text{ dominates } f)\}.$$

- (a) Show that the projections, the constants, the successor function and the function $\sup$ belong to C_0 ;
- (b) Show that for all $n \in \mathbb{N}$, C_n is closed under composition and contains ξ_n ;
- (c) Show that for all $n, k, x \in \mathbb{N}$, $\xi_n^k(x) \leq \xi_{n+1}(x + k)$;
- (d) Suppose that $g : \mathbb{N}^p \rightarrow \mathbb{N}$ and $h : \mathbb{N}^{p+2} \rightarrow \mathbb{N}$ belong to C_n , for a fixed $n \in \mathbb{N}$. Show that the function f defined by recursion on g and h belongs to C_{n+1} .

Hint: Show by induction that there is a relation of the form $f(x_1, \dots, x_p, y) \leq \xi_n^{k_1+yk_2}(\sup(x_1, \dots, x_p, y, B_1, B_2))$ for all $x_1, \dots, x_p, y$.

3. Show that $\bigcup_{n \in \mathbb{N}} C_n$ contains all primitive recursive functions.
4. Finally show that the Ackermann function ξ is not primitive recursive.
Hint: Towards a contradiction suppose that it is. Consider then the function $x \mapsto \xi(x, 2x)$ and show that there should exist $x, n, k \in \mathbb{N}$ such that $\xi(x, 2x) \leq \xi_{n+1}(x + k) < \xi(x, 2x)$.

Remark. *It can be shown (not easily) that the graph $G = \{(y, x, z) \in \mathbb{N}^3 \mid z = \xi(y, x)\}$ is in fact primitive recursive (see for example R. Cori and D. Lascar, Logique mathématique II, Chapter 5, exercise 11, p. 57). The Ackermann function is thus recursive since it can be defined as*

$$\xi(y, x) = \mu z \quad (y, x, z) \in G.$$

In general, indeed, one can show (easily) that a total function is recursive if and only if its graph is recursive. Finally notice that the Ackermann function is an example of a total function whose graph is primitive recursive but which nonetheless is not primitive recursive.