

Exercise Sheet n°11

- Let $\mathcal{L}_0$ be the first order language of arithmetic with non logical symbols $0, 1, +, \cdot, <$.
- Given a formula $\varphi(x)$ with a free variable x , an $\mathcal{L}_0$ -structure $\mathcal{M}$ and some $e \in \mathcal{M}$, we write $\mathcal{M} \models \varphi[e]$ to say that $\mathcal{M}$ models $\varphi(x)$ when x is interpreted as e .
- Let $\mathcal{N} = (\mathbb{N}, 0, 1, +, \cdot, <)$ be the standard model of arithmetic.
- We are only interested in distinguishing countable models of PA ‘up to isomorphism’. We can therefore restrict our attention to $\mathcal{L}_0$ -structures $\mathcal{M}$ whose domain is the set of natural numbers $\mathbb{N}$ and we can further take the interpretation of 0 and 1 to be standard too. So, without loss of generality, by a *model of PA* we henceforth mean a structure $\mathcal{M} = \langle \mathbb{N}, 0, 1, \oplus, \otimes, \otimes \rangle$ that satisfies the Peano axioms.
A model $\mathcal{M}$ of PA is called non standard if it not isomorphic to $\mathcal{N}$. We have already seen that there are continuum many ($2^{\aleph_0}$) such models.
- For any $n \in \mathbb{N}$ we denote by n its syntactic counterpart $\overbrace{1+\dots+1}^{n \text{ times}}$. In any model $\mathcal{M}$ of PA these terms are interpreted as 0, 1, $2^{\mathcal{M}} = 1 \oplus 1$, $3^{\mathcal{M}} = 1 \oplus 1 \oplus 1$. These elements are called standard elements. Notice that $n^{\mathcal{M}} = n$ holds for every $n \in \mathbb{N}$ *only* in the standard model.

Consider the following

Definition. A countable model $\mathcal{M} = \langle \mathbb{N}, 0, 1, \oplus, \otimes, \otimes \rangle$ of PA is called recursive if both binary functions $\oplus : \mathbb{N}^2 \rightarrow \mathbb{N}$, $\otimes : \mathbb{N}^2 \rightarrow \mathbb{N}$ are recursive and the binary relation $\otimes \subseteq \mathbb{N}^2$ is recursive.

The aim of this exercise sheet is to show

Tennenbaum’s Theorem.

The standard model is the only recursive model of PA.

Exercise 1: We consider a *recursive enumeration* of all Turing machines which compute partial functions from $\mathbb{N}$ to $\mathbb{N}$. According to this enumeration, we call $\mathcal{M}_k$ the k -th Turing machine. For all $k \in \mathbb{N}$, we let φ_k be the partial function from $\mathbb{N}$ to $\mathbb{N}$ computed by $\mathcal{M}_k$.

Show the following

Theorem 1. *There exist two nonempty recursively enumerable subsets A and B of $\mathbb{N}$ that are recursively inseparable, i.e. such that*

1. A and B are disjoint, and
2. there is no recursive set X such that $A \subseteq X$ and $X \cap B = \emptyset$.

Exercise 2:

Prove the following

Theorem 2 (Overspill). *Let $\varphi(x)$ be an arithmetic formula whose only free variable is x . Assume that $\mathcal{M}$ is a non standard model of PA such that for all n we have $\mathcal{M} \models \varphi[n]$. Then there is a non standard element $e \in |\mathcal{M}|$ such that $\mathcal{M} \models \varphi[e]$.*

Exercise 3:

Recall that a Δ_0 formula is a formula in which every quantification is bounded (these are the Δ_0^0 -rud from Exercise Sheet 6).

The n -th prime is denoted by $\pi(n)$, that is $\pi(0) = 2$, $\pi(1) = 3 \dots$. We have seen that the function $\pi : n \rightarrow \pi(n)$ is primitive recursive, so it can be represented in PA by a Σ_1^0 formula $P(x, y)$.

We will abbreviate $\exists y (P(k, y) \wedge y \otimes l = m)$ with $\pi(k) \otimes l = m$, meaning that the k -th prime multiplied by l equals m .

We make the following definition

Definition. We say that a subset $A \subseteq \mathbb{N}$

1. is coded in a model $\mathcal{M}$ if there exists a formula $\varphi(x, y)$ and some $a \in \mathcal{M}$ such that $A = \{n \in \mathbb{N} \mid \mathcal{M} \models \phi(n, [a])\}$,

2. is canonically coded in a model $\mathcal{M}$ if it is coded by the formula $\psi(x, y) : \exists z \pi(x) \otimes z = y$, i.e there exists $a \in \mathcal{M}$ such that $A = \{n \in \mathbb{N} \mid \mathcal{M} \models \exists z \pi(n) \otimes z = [a]\}$.

We need the following.

Theorem 3. Let $A(x, y)$ be a Δ_0 formula and $\mathcal{M}$ be a non standard model of PA. Then for all elements $b \in \mathcal{M}$ there exists an element $a \in \mathcal{M}$ such that for every $n \in \mathbb{N}$

$$\mathcal{M} \models \exists k \otimes [b] A(k, n) \leftrightarrow \exists z (\pi(n) \otimes z = [a])$$

We prove this in several steps:

1. We assume *without proof* that the following simple arithmetical truth is provable in PA, for every Δ_0 formula $A(x, y)$ and any $n \in \mathbb{N}$:

$$\forall b \exists a \forall u < n (\exists k < b A(k, u) \leftrightarrow \exists z (\pi(u) \cdot z = a)).$$

2. Show that for some non standard element e of $\mathcal{M}$ we have:

$$\mathcal{M} \models \forall b \exists a \forall u \otimes [e] (\exists k \otimes b A(k, u) \leftrightarrow \exists y (\pi(u) \otimes y = a)).$$

3. Conclude the proof of Theorem 3.

We recall the following basic fact

Theorem 4. The theory PA is Δ_0 complete, that is every Δ_0 formula which is true in the standard model is provable in PA.

We can now show that non standard elements can code non recursive sets.

Theorem 5. In any non standard model of PA, there is a non standard element which canonically codes a non recursive set.

Fix a non standard model $\mathcal{M}$ of PA. Here is how to proceed.

Exercise 4: By Theorem 1, let A and B be recursively enumerable sets that are recursively inseparable.

1. There exist two Δ_0 formulas $A(x, y)$ and $B(x, y)$ such that

$$A = \{m \mid \exists n A(m, n)\} \quad \text{and} \quad B = \{m \mid \exists n B(m, n)\}.$$

2. Show that there is a non standard $e \in \mathcal{M}$ such that

$$\mathcal{M} \models \forall x \otimes [e] \forall y \otimes [e] \forall z \otimes [e] \neg(A(x, y) \wedge B(x, z)).$$

Now consider the set

$$X = \{n \in \mathbb{N} \mid \mathcal{M} \models \exists y \otimes [e] A(n, y)\}.$$

3. Show that $A \subseteq X$.
4. Show that $B \cap X = \emptyset$.
5. Conclude the proof of Theorem 5.

We can now prove Tennenbaum's Theorem.

Exercise 5: Let $\mathcal{M} = \langle \mathbb{N}, 0, 1, \oplus, \otimes, \otimes \rangle$ be a non standard model of PA. Let X be a non recursive set canonically coded by some (non standard) element $a \in |\mathcal{M}| = \mathbb{N}$.

1. Observe that $\mathcal{M}$ satisfies the following formulas

- $\forall x \forall y (y \neq 0 \rightarrow \exists! q \exists! r (r \otimes y \wedge x = (q \otimes y) \oplus r))$ (Euclidean division);
- for all n , $\forall x (\pi(n) \otimes x = \underbrace{x \oplus \dots \oplus x}_{\pi(n) \text{ times}})$;
- for all n , $\forall x (x \otimes \pi(n) \leftrightarrow x = 0 \vee x = 1 \vee \dots \vee x = \underbrace{1 \oplus \dots \oplus 1}_{\pi(n) \text{ times}})$.

2. Towards a contradiction suppose that $\oplus : \mathbb{N}^2 \rightarrow \mathbb{N}$ is recursive and design a decision procedure for X .
3. Conclude!