

1. a) We only need to show that for $v \in K_\infty^\times$ and $x \in K^\times$, we have $\lambda(x, v)_i \neq 0$ for all i . As σ_i has trivial kernel for all i , this is immediate.
- b) Assume $v \in K_\infty^\times$ and $x \in K^\times$ satisfy $\lambda(x, v) = v$. Then $\sigma_i(x) = 1$ for all $1 \leq i \leq r$ and, by injectivity of σ_i , it follows that $x = 1$.
- c) Let $\ell_1, \ell_2 \in HM$ and let $h_1, h_2 \in H$, $m_1, m_2 \in M$ such that $\ell_i = h_i m_i$, then

$$\ell_1 \ell_2^{-1} = h_1 m_1 m_2^{-1} h_2^{-1} = h_1 h_2^{-1} m_1 m_2^{-1} \in HM,$$

since $K^\times$ is abelian. Since $1 = 1 \cdot 1 \in HM$, we know that HM is non-empty and therefore a subgroup.

The map $f: H \times M \rightarrow HM$ given by $(h, m) \mapsto hm^{-1}$ is a surjective group homomorphism (since $K^\times$ is abelian) with kernel

$$\begin{aligned} \ker f &= \{(h, m) \in H \times M : hm^{-1} = 1\} \\ &= \{(h, m) \in H \times M : h = m\} \cong H \cap M = \{1\}. \end{aligned}$$

Hence f is injective and, therefore, an isomorphism.

- d) Let $\psi: E/M \rightarrow E/HM$ denote the canonical projection. Let $O_1, O_2 \in E/M$ such that $\psi(O_1) = \psi(O_2)$. Equivalently, there are $v_i \in O_i$ and $\ell \in HM$ such that $v_2 = \lambda(\ell, v_1)$. Write $\ell = hm$ for $h \in H$ and $m \in M$. Then

$$\lambda(\ell, x_1) = \lambda(h, \lambda(m, x_1))$$

and $x_1 \in O_1$ if and only if $\lambda(m, x_1) \in O_1$. Therefore, we have shown that $\psi(O_1) = \psi(O_2)$ if and only if there are $x_i \in O_i$ and $h \in H$ such that $x_2 = \lambda(h, x_1)$.

It remains to show that the fibers of ψ form a disjoint union of M -orbits parametrized by H . To this end suppose that $O_1, O_2 \in E/M$ are such that there exist $x_i \in O_i$ and $h \in H \setminus \{1\}$ such that $x_2 = \lambda(h, x_1)$. Suppose that $y \in O_1 \cap O_2$, then there are $m_i \in M$ such that $y = \lambda(m_1, x_1) = \lambda(m_2, x_2)$. Hence

$$x_1 = \lambda(m_1^{-1}, y) = \lambda(m_1^{-1} m_2, x_2) = \lambda(m_1^{-1} m_2 h, x_1)$$

and therefore $h = m_1 m_2^{-1}$ by freeness. As $H \cap M = \{1\}$, this is absurd.

- e) Let $y = \text{Nr}_{K/\mathbb{Q}}(x)x^{-1}$ and consider $z = x^{1-d}y$. By Dirichlet's unit theorem, there are $A > 0$ depending only on K and $u \in \mathcal{O}_K^\times$ such that

$$\forall 1 \leq i \leq r \quad -A \leq d_i(\log|\sigma_i(z)| - d \log|\sigma_i(u)|) \leq A.$$

Since

$$\forall 1 \leq i \leq r \quad \sigma_i(zu^{-d}) = \text{Nr}_{K/\mathbb{Q}}(x)\sigma_i(x^{-d}u^{-d}),$$

it follows that

$$\forall 1 \leq i \leq r \quad e^{-A/d}|\sigma_i(xu)| \leq \text{Nr}_{K/\mathbb{Q}}(x) \leq e^{A/d}|\sigma_i(xu)|.$$

The details are left to the reader.

2. This can be found in the lecture notes.
3. a) This can be found in the lecture notes.

b) If p splits,

$$\prod_{\mathfrak{p}|(p)} \left(1 - \frac{1}{N(\mathfrak{p})^s}\right)^{-1} = \left(1 - \frac{1}{p^s}\right)^{-2} = \left(1 - \frac{1}{p^s}\right)^{-1} \left(1 - \frac{\chi_K(p)}{p^s}\right)^{-1},$$

if p is inert,

$$\prod_{\mathfrak{p}|(p)} \left(1 - \frac{1}{N(\mathfrak{p})^s}\right)^{-1} = \left(1 - \frac{1}{p^{2s}}\right)^{-1} = \left(1 - \frac{1}{p^s}\right)^{-1} \left(1 - \frac{\chi_K(p)}{p^s}\right)^{-1},$$

and, finally, if p is ramified,

$$\prod_{\mathfrak{p}|(p)} \left(1 - \frac{1}{N(\mathfrak{p})^s}\right)^{-1} = \left(1 - \frac{1}{p^s}\right)^{-1} = \left(1 - \frac{1}{p^s}\right)^{-1} \left(1 - \frac{\chi_K(p)}{p^s}\right)^{-1}.$$

Altogether, this gives

$$\zeta_K(s) = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1} \cdot \prod_{p \text{ prime}} \left(1 - \frac{\chi_K(p)}{p^s}\right)^{-1} = \zeta(s) L(s, \chi_K),$$

as we wanted to show.

4. a) We note that $|\psi(n)| \leq 1$ and hence, for $\operatorname{Re} s > 1$, the Euler product formula follows in the same way as for the Riemann ζ -function from the fact that ψ is completely multiplicative, i.e., if $m, n \in \mathbb{N}$, then $\psi(mn) = \psi(m)\psi(n)$.

b) We have

$$\prod_p \left(1 - \frac{\psi_0(p)}{p^s}\right)^{-1} = \prod_{(p,q)=1} \left(1 - \frac{1}{p^s}\right)^{-1} = \prod_{(p,q) \neq 1} \left(1 - \frac{1}{p^s}\right) \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} = \zeta(s) \prod_{p|q} \left(1 - \frac{1}{p^s}\right).$$

c) Note that

$$\sum_{n=1}^q \psi(n) = \sum_{(n,q)=1} \chi(n) = 0,$$

and therefore $\Psi(x) = \sum_{n \leq x} \psi(n) \ll \varphi(q)$. Using Abel summation, we have for all $0 < T_1 < T_2$ that

$$\sum_{T_1 < n \leq T_2} \frac{\psi(n)}{n^s} = \left[\frac{\Psi(\xi)}{\xi^s} \right]_{T_1}^{T_2} + s \int_{T_1}^{T_2} \frac{\Psi(\xi)}{\xi^{s+1}} s.$$

Since $\Psi(\xi) \ll_q 1$, the right hand side converges uniformly on compact subsets of $\{s: \operatorname{Re} s > 0\}$.

d) We do not provide a solution to this exercise yet.

- e) Note that G_K is a finite product, hence holomorphic on $\{s: \operatorname{Re} s > 0\}$. Similarly, $L(\psi_0, s)$ is a product of a holomorphic function and a meromorphic function on $\{s: \operatorname{Re} s > 0\}$ with a simple pole at $s = 1$. Recall that for $\psi \neq \psi_0$, the function $L(\psi, s)$ is holomorphic on $\{s: \operatorname{Re} s > 0\}$. Hence the function

$$\zeta_K(s) = G_K(s) L(\psi_0, s) \prod_{\psi \neq \psi_0} L(\psi, s)$$

is meromorphic with a unique simple pole at $s = 1$ if and only if $L(\psi, 1) \neq 0$ for all $\psi \neq \psi_0$. Since, as shown in class, the Dedekind ζ -function ζ_K has a simple pole at $s = 1$, the claim follows.

f) We show that $L(\psi, s)$ admits an Euler product. We recall from complex analysis that an infinite product of non-zero numbers $(z_n)_{n \in \mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$ converges (in $\mathbb{C} \setminus \{0\}$) if and only if the series $\sum_{n=1}^{\infty} \log z_n$ converges, where $\log z_n$ is the logarithm for the principal branch. Moreover, the product converges absolutely if and only if the series converges absolutely and, in fact, this is equivalent to the convergence of the series $\sum_{n=1}^{\infty} |z_n - 1|$. In this case, we have

$$\prod_n z_n = \exp \left(\sum_{n \in \mathbb{N}} \log(z_n) \right).$$

We apply this to the sequence $z_p(s) = 1 - \frac{\psi(p)}{p^s}$, where $\operatorname{Re} s > 1$ and where we use the primes to index the sequence (as opposed to using the natural number n to index the n -th prime).

Hence, in order to prove that the product

$$P(s) = \prod_p \left(1 - \frac{\psi(p)}{p^s} \right)^{-1}$$

converges absolutely, it suffices to show that the series $\sum_p \frac{\psi(p)}{p^s}$ converges absolutely. To this end, we use the convergence of the Riemann ζ -function for $\operatorname{Re} s > 0$ to obtain

$$\sum_p \left| \frac{\psi(p)}{p^s} \right| \leq \sum_{n \in \mathbb{N}} \frac{1}{n^{\operatorname{Re} s}} = \zeta(\operatorname{Re} s) < \infty.$$

Next we argue that $P(s)$ is holomorphic on $\{\operatorname{Re} s > 1\}$. To this end it suffices to prove that $P(s) = L(\psi, s)$, of which we know holomorphicity by Abel summation. Given $N \in \mathbb{N}$, let $\mathcal{P}(N)$ denote the collection of the first N primes and let $\mathbb{N}(N)$ denote the set of natural numbers whose prime factorization only involves primes in $\mathcal{P}(N)$.

Then

$$\prod_{p \in \mathcal{P}(N)} \left(1 - \frac{\psi(p)}{p^s} \right)^{-1} = \prod_{p \in \mathcal{P}(N)} \sum_{\ell \in \mathbb{N}} \frac{\psi(p)^\ell}{p^{\ell s}} = \sum_{n \in \mathbb{N}(N)} \frac{\psi(n)}{n^s}.$$

Therefore

$$\left| L(\psi, s) - \prod_{p \in \mathcal{P}(N)} \left(1 - \frac{\psi(p)}{p^s} \right)^{-1} \right| \leq \sum_{n > \max \mathcal{P}(N)} \frac{1}{n^{\operatorname{Re} s}} \xrightarrow{N \rightarrow \infty} 0$$

and thus $P(s) = L(\psi, s)$, as desired.

Since $L(\psi, s) = P(s)$ is holomorphic on $\{\operatorname{Re} s > 1\}$ and $P(s)$ does not vanish, $L(\psi, s)$ admits an analytic logarithm on $\{\operatorname{Re} s > 1\}$. Moreover, we have that

$$L(\psi, s) = \exp \left(- \sum_p \log \left(1 - \frac{\psi(p)}{p^s} \right) \right)$$

and hence a choice of a logarithm at $\operatorname{Re} s > 1$ is given by

$$\log L(\psi, s) = - \sum_p \log \left(1 - \frac{\psi(p)}{p^s} \right).$$

As shown in class, the right-hand side satisfies

$$\sum_p \log \left(1 - \frac{\psi(p)}{p^s} \right) = \sum_p \frac{\psi(p)}{p^s} + \underbrace{\sum_p \sum_{\ell \geq 2} \frac{\psi(p^\ell)}{\ell p^{\ell s}}}_{=g_\psi}$$

with g_ψ converging uniformly on compact subsets of $\{\operatorname{Re} s > 1/2\}$. Hence the claim.

- g) In order to prove that there are infinitely many primes with residue $a \bmod q$, it suffices to show that

$$\sum_{p \equiv a} \frac{1}{p^s} \xrightarrow{s \downarrow 1} \infty.$$

To this end, we note that by the orthogonality relations

$$\begin{aligned} \sum_{p \equiv a} \frac{1}{p^s} &= \sum_p \mathbf{1}_{a+q\mathbb{Z}}(p) \frac{1}{p^s} = \sum_p \mathbf{1}_{1+q\mathbb{Z}}(a^{-1}p) \frac{1}{p^s} \\ &= \sum_p \left(\frac{1}{\varphi(q)} \sum_\psi \psi(a^{-1}p) \right) \frac{1}{p^s} = \frac{1}{\varphi(q)} \sum_\psi \psi(a^{-1}) \sum_p \frac{\psi(p)}{p^s} \\ &= \frac{1}{\varphi(q)} \sum_\psi \psi(a^{-1}) \log L(\psi, s) - \frac{1}{\varphi(q)} \sum_\psi \psi(a^{-1}) g_\psi(s). \end{aligned}$$

The function

$$s \mapsto \frac{1}{\varphi(q)} \sum_\psi \psi(a^{-1}) g_\psi \quad (s > 1)$$

admits a holomorphic continuation to $\{\operatorname{Re} s > 3/4\}$ and hence it remains bounded as $s \downarrow 1$.

For the first sum, we note that

$$\sum_\psi \psi(a^{-1}) L(\psi, s) = L(\psi_0, s) + \sum_{\psi \neq \psi_0} \psi(a^{-1}) L(\psi, s)$$

and the sum on the right hand side remains bounded as $s \downarrow 1$ because of part (e). On the other hand, we know that $L(\psi_0, s) \rightarrow \infty$ as $s \downarrow 1$ and hence the claim.