1. a) First of all, if d is square-free we must have $d \equiv 1, 2, 3 \mod 4$ since $d \equiv 0 \mod 4$ would imply that 2^2 divides d.
   b) Now we can consider the two cases:
      (i) $d \equiv 2, 3 \mod 4$. The discriminant of the basis $(1, \sqrt{d})$ is
$$\left(\det \begin{pmatrix} 1 & \sqrt{d} \\ 1 & -\sqrt{d} \end{pmatrix}\right)^2 = 4d.$$
      Let $\mathcal{B}$ be a $\mathbb{Z}$-basis of $\mathcal{O}_K$, and let M be an integer matrix such that $\mathcal{B}M = (1, \sqrt{d})$. We have
$$\operatorname{disc}(\mathcal{O}_K) = \frac{\operatorname{disc}(1, \sqrt{d})}{\det(M)^2} = \frac{4d}{\det(M)^2}.$$
      Since d is square-free the only possibilities are $|\det(M)| = 1$ and $|\det(M)| = 2$. In the latter case we would have $\operatorname{disc}(\mathcal{O}_K) = d \equiv 2, 3 \mod 4$, which is impossible since, as we have shown on Sheet 8 in Exercise 2 that the discriminant of the ring of integers is always 0 or 1 mod 4. Therefore we must have $|\det(M)| = 1$, meaning that $(1, \sqrt{d})$ is a $\mathbb{Z}$-basis of $\mathcal{O}_K$ and therefore $\mathcal{O}_K = \mathbb{Z}[\sqrt{d}]$.
      (ii) $d \equiv 1 \mod 4$. We observe that $\frac{1+\sqrt{d}}{2} \in \mathcal{O}_K$ since it is a root of the integer polynomial $p(x) = x^2 - x - \frac{d-1}{4}$. We leave the verification to the reader. The discriminant of the $\mathbb{Q}$-basis $\left(1, \frac{1+\sqrt{d}}{2}\right)$ of K is
$$\left(\det \begin{pmatrix} 1 & \frac{1+\sqrt{d}}{2} \\ 1 & \frac{1-\sqrt{d}}{2} \end{pmatrix}\right)^2 = d.$$
      Let $\mathcal{B}$ be a $\mathbb{Z}$-basis of $\mathcal{O}_K$, and let M be an integer matrix such that $\mathcal{B}M = \left(1, \frac{1+\sqrt{d}}{2}\right)$. We have
$$\operatorname{disc}(\mathcal{O}_K) = \frac{\operatorname{disc}\left(1, \frac{1+\sqrt{d}}{2}\right)}{\det(M)^2} = \frac{d}{\det(M)^2}.$$
      Since d is square-free the only possibility is $|\det(M)| = 1$, meaning that $\left(1, \frac{1+\sqrt{d}}{2}\right)$ is a $\mathbb{Z}$-basis of $\mathcal{O}_K$ and therefore $\mathcal{O}_K = \mathbb{Z}\left[\frac{1+\sqrt{d}}{2}\right]$.
2. a) Let $K_1 = \mathbb{Q}(\sqrt{-m})$ and $K_2 = \mathbb{Q}(\sqrt{-n})$. The extensions $K_1/\mathbb{Q}$ and $K_2/\mathbb{Q}$ are of degree 2 and hence Galois, with Galois group isomorphic to $\mathbb{Z}/2\mathbb{Z}$. The composite field K_1K_2 is K, and the intersection $K_1 \cap K_2$ is $\mathbb{Q}$ since m, n are distinct and squarefree. Therefore the extension $K/\mathbb{Q}$ is Galois of degree 4, and we have
$$\operatorname{Gal}(K/\mathbb{Q}) \cong \operatorname{Gal}(K_1/\mathbb{Q}) \times \operatorname{Gal}(K_2/\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$
   b) Proper subfields of K are in one-to-one correspondence with non-trivial subgroups of the Galois group. Since $\operatorname{Gal}(K/\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, it has three proper non-trivial subgroups, all isomorphic to $\mathbb{Z}/2\mathbb{Z}$ and generated by the pairs

$(1,0),(0,1),(1,1)$ in $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$. Therefore there are exactly three subfields L that satisfy $\mathbb{Q}\subset L\subset K$ with proper inclusions. We can find them explicitly: they are $\mathbb{Q}(\sqrt{-m})$, $\mathbb{Q}(\sqrt{-n})$ and $\mathbb{Q}(\sqrt{mn})$. They are all non-isomorphic, and the last one is the only real subfield.

c) We recall that $\mathcal{O}_L$ is the set of elements in L that are integral over $\mathbb{Z}$, and $\mathcal{O}_K$ is the set of elements in K that are integral over $\mathbb{Z}$. Since $L\subseteq K$, it is immediate that $\mathcal{O}_L=L\cap\mathcal{O}_K$. Moreover, if $x\in\mathcal{O}_L$ has an inverse in $\mathcal{O}_L$ then this inverse is also in $\mathcal{O}_K$, hence $\mathcal{O}_L^\times\subseteq L\cap\mathcal{O}_K^\times$. Conversely, if $x\in L\cap\mathcal{O}_K$ has an inverse in $\mathcal{O}_K$, since L is a field this inverse must lie in L and, since elements of $\mathcal{O}_K$ are integral, hence in $\mathcal{O}_L$, providing the inclusion $L\cap\mathcal{O}_K^\times\subseteq\mathcal{O}_L^\times$.

d) By Dirichlet's unit theorem, an element in $\mathcal{O}_K^\times$ is a root of unity if and only if it is in the kernel of the logarithmic embedding. Therefore, for $\varepsilon\in\mathcal{O}_K^\times$, we have $\frac{\varepsilon}{\sigma(\varepsilon)}\in\mu_K$ if and only if

$$\forall\tau\in\mathrm{Hom}_{\mathbb{Q}}(K,\mathbb{C})\qquad\left|\tau\left(\frac{\varepsilon}{\sigma(\varepsilon)}\right)\right|=1\ .$$

This is true if and only if $|\tau(\varepsilon)|=|\tau(\sigma(\varepsilon))|$ for all $\tau\in\mathrm{Hom}_{\mathbb{Q}}(K,\mathbb{C})$. Since $K/\mathbb{Q}$ is Galois we have $\mathrm{Hom}_{\mathbb{Q}}(K,\mathbb{C})=\mathrm{Gal}(K/\mathbb{Q})$ and, since the Galois group is commutative, the condition is equivalent to $|\tau(\varepsilon)|=|\sigma(\tau(\varepsilon))|\ \forall\tau\in\mathrm{Gal}(K/\mathbb{Q})$. Now one shows that the assumptions imply that σ must send 1 to 1, $\sqrt{mn}$ to $\sqrt{mn}$, $\sqrt{-m}$ to $-\sqrt{-m}$, and $\sqrt{-n}$ to $-\sqrt{-n}$. One checks that this implies that σ agrees with the restriction to K of the conjugation in $\mathbb{C}$ and therefore preserves the absolute value. We leave the details and the remaining parts to the reader.

e) Let $\mu_K^2=\{\zeta\in\mathcal{O}_K^\times\colon\exists\xi\in\mu_K\,\zeta=\xi^2\}$. Then μ_K^2 is the image of the squaring homomorphism $\mu_K\to\mu_K$, $\zeta\mapsto\zeta^2$, hence a subgroup of μ_K. We denote by $\Phi\colon\mathcal{O}_K^\times\to\mu_K/\mu_K^2$ the composition of ϕ with the canonical projection. Then $\ker\Phi=\mu_K\mathcal{O}_L^\times$. We sketch a proof of $\ker\Phi\subseteq\mu_K\mathcal{O}_K^\times$ and leave the opposite direction to you. Suppose that $\varepsilon\in\ker\Phi$ and write $\varepsilon=r\zeta$ with $r\in(0,\infty)$ and $\zeta\in\mathbb{S}^1$. One checks that $\psi(\varepsilon)=\zeta^2$. Since $\zeta^2\in\mu_K^2$. One checks that this implies $\zeta\in\mu_K$ and, hence, $r\in\mathcal{O}_K^\times\cap L_{\mathbb{R}}=\mathcal{O}_{L_{\mathbb{R}}}^\times$.

Now one uses the isomorphism theorems to deduce that

$$[\mathcal{O}_K^\times\colon\mu_K\mathcal{O}_L^\times]\leqslant|\mu_K/\mu_K^2|$$

and, therefore, it remains to show that $[\mu_K\colon\mu_K^2]\leqslant 2$. Then one shows that μ_K is cyclic and, using the classification of finite cyclic groups, deduces that μ_K^2 has index one in μ_K if N is odd and index two if N is even.

3. a) Note that for any $x,y\in K^\times$ totally positive, the product xy is totally positive. This implies that the totally positive elements in $K^\times$ form a subgroup of $K^\times$, which then implies that $\mathcal{P}_K^+$ is a subgroup of $\mathcal{P}_K$ and, hence, of $\mathcal{F}_K$. We skip the details of this bit of the proof. It remains to prove finiteness of the *narrow class group* $\mathrm{Cl}^+(K)$.

We let $q\colon\mathrm{Cl}^+(K)\to\mathrm{Cl}(K)$ denote the canonical projection, which is well-defined since $\mathcal{P}_K^+\subseteq\mathcal{P}_K$. Let $H=\ker q$, $\mathrm{Cl}(K)\cong\mathrm{Cl}^+(K)/H$ since q is surjective and, since $\mathrm{Cl}(K)$ is finite, it suffices to show that H is finite.

Our goal is to express H as a quotient of finite groups. Let $\psi\colon K^\times\to\mathrm{Cl}^+(K)$ denote the map $x\mapsto(x\mathcal{O}_K)\mathcal{P}_K^+$. Then $q\circ\psi$ maps $K^\times$ to $\mathcal{P}_K^+$, i.e., $\psi(K^\times)\subseteq H$. On the other hand, let $\mathfrak{f}\lhd K$ be a fractional $\mathcal{O}_K$-ideal such that $\mathfrak{f}\mathcal{P}_K^+\in H$. Since, by assumption, $\mathfrak{f}\mathcal{P}_K=\mathcal{P}_K$, i.e., since $\mathfrak{f}\mathcal{P}_K^+\in H$, we know that $\mathfrak{f}$ is a principal

fractional ideal, i.e., there exists $x \in K^\times$ such that $\mathfrak{f} = x\mathcal{O}_K$. In particular, we find that $\psi(K^\times) = H$.
Now denote by K^+ the totally positive elements in K and note that for any $x \in K^\times$ and for any $y \in K^+$ we have that

$$\psi(xy) = (xy\mathcal{O}_K)\mathcal{P}_K + = (x\mathcal{O}_K)(y\mathcal{O}_K)\mathcal{P}_K^+ = (x\mathcal{O}_K)\mathcal{P}_K^+ = \psi(x),$$

that is, ψ factors as

$$\begin{tikzcd} K^\times \arrow[r, "\psi"] \arrow[d, two heads] & H \\ K^\times/K^+ \arrow[ur, dashed, "\Psi"'] & \end{tikzcd}$$

Next one show that $\ker\Psi = \mathcal{O}_K^\times K^+$ using that the generator of a principal fractional ideal is unique up to multiplication by $\mathcal{O}_K^\times$. In the next step one shows that $\mathcal{O}_K^\times K^+/K^+ \cong \mathcal{O}_K^\times/\mathcal{O}_{K+}^\times$, where the main step involves checking that if $\varepsilon_1, \varepsilon_2 \in \mathcal{O}_K^\times$ satisfy $\varepsilon_1 K^+ = \varepsilon_2 K^+$, then $\varepsilon_1\mathcal{O}_{K+}^\times = \varepsilon_2\mathcal{O}_{K+}^\times$. Using the appropriate isomorphism theorem, it follows that

$$|H| = \frac{[K^\times : K^+]}{[\mathcal{O}_K^\times : \mathcal{O}_{K+}^\times]}.$$

Hence, in the light of 3b, it remains to show that K^+ has finite index in $K^\times$.
To this end, we recall that $\sigma_\infty(K)$ is dense in $\mathbb{R}^s$, thus one can show that for all $1 \leqslant i \leqslant d$ there exists an element $x_i \in K^\times$ such that

$$\forall 1 \leqslant j \leqslant d \quad \sigma_j(x_i) < 0 \iff j = i.$$

In particular, given $I \subseteq \{1, \dots, d\}$, let

$$y_I = \prod_{i \in I} x_i^{-1},$$

then

$$K^\times \subseteq \bigcup_{I \subseteq \{1,\dots,d\}} y_I K^+$$

and, therefore, $[K^\times : K^+] \leqslant 2^d$.
The details are left to the reader.

b) Let $\varphi\colon \mathcal{O}_K^\times \to \mathcal{O}_K^\times$ denote the squaring homomorphism, i.e., for all $\varepsilon \in \mathcal{O}_K^\times$ we have that

$$\varphi(\varepsilon) = \varepsilon^2.$$

Then

$$\mathrm{Log}_\infty(\mathrm{im}\varphi) = \left\{\mathrm{Log}_\infty(\varepsilon^2) \colon \varepsilon \in \mathcal{O}_K^\times\right\} = 2\mathrm{Log}_\infty(\mathcal{O}_K^\times)$$

and, hence, by Dirichlet's unit theorem $\mathrm{Log}_\infty(\mathrm{im}\varphi) \subseteq \mathbb{Z}^{r-1}$. In particular, we find that

$$\mathbb{Z}^{r-1} \cong \mathrm{Log}_\infty(\mathrm{im}\varphi) \subseteq \mathrm{Log}_\infty(\mathcal{O}_{K+}^\times) \subseteq \mathrm{Log}_\infty(\mathcal{O}_K^\times) \cong \mathbb{Z}^{r-1};$$

therefore proving that $\mathrm{Log}_\infty(\mathcal{O}_{K+}^\times) \cong \mathbb{Z}^{r-1}$ by the structure theory for $\mathbb{Z}$-modules. In particular, it suffices to show that $\ker \mathrm{Log}_\infty|_{\mathcal{O}_{K+}^\times}$ is trivial. Since

$$\ker \mathrm{Log}_\infty|_{\mathcal{O}_{K+}^\times} = \mathcal{O}_{K+}^\times \cap \ker \mathrm{Log}_\infty|_{\mathcal{O}_K^\times} = \mathcal{O}_{K+}^\times \cap \mu_K,$$

we just have to show that 1 is the only totally positive root of unity. But, in fact, $\{\pm 1\}$ are the only real roots of unity and, since every relevant embedding is $\mathbb{Q}$-linear, 1 is the only totally positive real root of unity.

c) Let
$$E = (-\infty, 0)^{\Sigma} \times (0, \infty)^{\mathrm{Hom}_{\mathbb{Q}}(K,\mathbb{R}) \smallsetminus \Sigma}.$$
Our goal is to show that
$$\emptyset \neq \mathrm{Log}_{\infty}(\mathcal{O}_{K+}^{\times}) \cap E.$$
We denote by $\mathbf{1} \in \mathbb{R}^d$ the vector with constant coordinate equal to 1 and let $H = \mathbf{1}^{\perp}$. We denote by $\|\cdot\|$ the norm induced by the standard inner product on $\mathbb{R}^d$. Given a subspace W of $\mathbb{R}^d$, a vector $\mathbf{w} \in W$, and a radius $r > 0$, we denote
$$B_r^W(\mathbf{w}) = \{\mathbf{w}' \in W \colon \|\mathbf{w} - \mathbf{w}'\| < r\},$$
i.e., $B_r^W(\mathbf{w})$ is the ball of radius r inside W around $\mathbf{w}$, where the metric is the one induced by the restriction of the inner product to W. Given $\mathbf{v} \in \mathbb{R}^d$, we denote
$$B_r(\mathbf{v}) = B_r^{\mathbb{R}^d}(\mathbf{v}).$$
We note that $B_r^W(\mathbf{w}) = B_r(\mathbf{w}) \cap W$.
Claim 1. For every $r > 0$ there exists $\mathbf{v} \in H$ such that
$$B_r^H(\mathbf{v}) \subseteq E.$$
Proof of Claim 1. Since $B_r^H(\mathbf{v}) = B_r(\mathbf{v}) \cap H$, it suffices to show that there exists $\mathbf{v} \in H$ such that $B_r(\mathbf{v}) \subseteq E$. In what follows, we denote by $\mathbf{I}_{\Sigma} \in \mathbb{R}^d = \mathbb{R}^{\mathrm{Hom}_{\mathbb{Q}}(K,\mathbb{R})}$ the vector
$$\forall \sigma \in \mathrm{Hom}_{\mathbb{Q}}(K,\mathbb{R}) \quad \mathbf{I}_{\Sigma}(\sigma) = \begin{cases} -|\Sigma|^{-1} & \sigma \in \Sigma, \\ |\mathrm{Hom}_{\mathbb{Q}}(K,\mathbb{R}) \smallsetminus \Sigma|^{-1} & \sigma \notin \Sigma. \end{cases}$$
Since by assumption $\Sigma \notin \{\emptyset, \mathrm{Hom}_{\mathbb{Q}}(K,\mathbb{R})\}$, the vector $\mathbf{I}_{\Sigma}$ is well-defined, contained in E, and
$$\sum_{\sigma \in \mathrm{Hom}_{\mathbb{Q}}(K,\mathbb{R})} \mathbf{I}_{\Sigma}(\sigma) = -\frac{1}{|\Sigma|} \sum_{\sigma \in \Sigma} 1 + \frac{1}{|\mathrm{Hom}_{\mathbb{Q}}(K,\mathbb{R}) \smallsetminus \Sigma|} \sum_{\sigma \in \mathrm{Hom}_{\mathbb{Q}}(K,\mathbb{R}) \smallsetminus \Sigma} 1 = 0$$
implies that $\mathbf{I}_{\Sigma} \in H$. We denote $M = \max\{|\Sigma|, |\mathrm{Hom}_{\mathbb{Q}}(K,\mathbb{R}) \smallsetminus \Sigma|\}$.
Let now $r > 0$ arbitrary and set $\mathbf{v} = 2rM\mathbf{I}_{\Sigma}$, then $\mathbf{v} \in H$ since H is a vector space. We leave it to you to check that $B_r(\mathbf{v}) \subseteq E$.
Now recall that $\mathrm{Log}_{\infty}(\mathcal{O}_K^{\times})$ is a lattice in H and, hence, so is $\mathrm{Log}_{\infty}(\mathcal{O}_{K+}^{\times})$. Fix a $\mathbb{Z}$-basis $\mathcal{B} = (\mathbf{v}_1, \ldots, \mathbf{v}_{d-1}) \in H^{d-1}$ of $\mathrm{Log}_{\infty}(\mathcal{O}_{K+}^{\times})$. We denote by $\|\cdot\|_{\mathcal{B}}$ the sup-norm on H defined with respect to $\mathcal{B}$ and we recall that. Given $r > 0$ and $\mathbf{v} \in H$, let $B_r^{\mathcal{B}}(\mathbf{v})$ denote the r-ball around $\mathbf{v}$ with respect to the metric induced by $\|\cdot\|_{\mathcal{B}}$. Recall that
$$\mathcal{P} = \left\{ \sum_{i=1}^{d-1} t_i \mathbf{v}_i \colon \forall 1 \leqslant i < d \quad -\frac{1}{2} \leqslant t_i < \frac{1}{2} \right\}$$
is a fundamental domain for $\mathrm{Log}_{\infty}(\mathcal{O}_{K+}^{\times}) \curvearrowright H$ and note that
$$B_{1/2}^{\mathcal{B}}(\mathbf{0}) \subseteq \mathcal{P} \subseteq \overline{B_{1/2}^{\mathcal{B}}(\mathbf{0})}.$$

Claim 2. For every $\mathbf{v} \in H$, the set $B_1^{\mathcal{B}}(\mathbf{v}) = B_1^{\mathcal{B}}(\mathbf{0}) + 0$ intersects $\mathrm{Log}_\infty(\mathcal{O}_{K+}^\times)$ non-trivially.
Proof of Claim 2. Since $\mathcal{P}$ is a fundamental domain, there exists $\ell \in \mathrm{Log}_\infty(\mathcal{O}_{K+}^\times)$ such that $\mathbf{v} \in \ell + \mathcal{P}$. One needs to show that $\|\mathbf{v} - \ell\|_{\mathcal{B}} < 1$. The details are left to you.
Now one uses the equivalence of norms on H to choose $r^* > 0$ such that for all $\mathbf{v} \in H$ we have

$$B_1^{\mathcal{B}}(\mathbf{v}) \subseteq B_{r^*}^{H}(\mathbf{v}).$$

Using Claim 1, there is $\mathbf{v} \in H$ such that $B_{r^*}^{H}(\mathbf{v}) \subseteq E$, hence $B_1^{\mathcal{B}}(\mathbf{v}) \subseteq E$ and, as of Claim 2, $\mathrm{Log}_\infty(\mathcal{O}_{K+}^\times)$ intersects E non-trivially, which completes the proof. The details are left to the reader.

4. a) We prove that

$$\sum_{y<n\leqslant x} a_n \psi(n) = A(x)\psi(x) - A(y)\psi(y) - \int_y^x A(\xi)\psi'(\xi)\mathrm{d}\xi.$$

Given $\alpha \in \mathbb{R}$, let $[\alpha]$ denote the largest integer bounding α from below. Note that we can assume without loss of generality that $[y] + 1 \leqslant [x]$, since A is constant on $[n, n+1)$ for $n \in \mathbb{Z}$.
We have

$$\begin{aligned}\sum_{y<n<x} a_n \psi(n) &= \sum_{y<n\leqslant [x]} \big(A(n) - A(n-1)\big)\psi(n) \\ &= \sum_{[y]+1\leqslant n\leqslant [x]} A(n)\psi(n) - \sum_{[y]\leqslant n\leqslant [x]-1} A(n)\psi(n-1) \\ &= A(x)\psi([x]) - A(y)\psi([y]+1) - \sum_{[y]\leqslant n<[x]} A(n)\big(\psi(n+1) - \psi(n)\big) \\ &= A(x)\psi([x]) - A(y)\psi([y]+1) - \int_{[y]+1}^{[x]} A(\xi)\psi'(\xi)\mathrm{d}\xi.\end{aligned}$$

In order to conlude, one uses once more that A is constant on intervals of the form $[n, n+1)$ with $n \in \mathbb{Z}$.

b) We apply Abel summation with the constant sequence $a_n = 1$, so that $A(\xi) = [\xi]$, and $\psi(\xi) = \xi^{-s}$ with $\mathrm{Re}\, s > 1$ to obtain that

$$\begin{aligned}\sum_{1/2<n\leqslant N} \frac{1}{n^s} &= N^{1-s} - s\int_1^N \frac{[\xi]}{\xi^{s+1}}\mathrm{d}\xi \\ &= N^{1-s} + s\int_1^N \frac{\xi - [\xi]}{\xi^{s+1}}\mathrm{d}\xi - s\int_1^N \frac{\mathrm{d}\xi}{\xi^s} \\ &= \frac{1-2s}{1-s}N^{1-s} + \frac{1}{1-s} - 1 + s\int_1^N \frac{\xi - [\xi]}{\xi^{s+1}}\mathrm{d}\xi \\ &\xrightarrow{N\to\infty} \frac{1}{1-s} - 1 + s\int_1^\infty \frac{\xi - [\xi]}{\xi^{s+1}}\mathrm{d}\xi.\end{aligned}$$

One checks that the function

$$s \mapsto \int_1^\infty \frac{\xi - [\xi]}{\xi^{s+1}}\mathrm{d}\xi$$

is holomorphic on $\{s\colon \operatorname{Re}(s) > 0\}$ using the characterization of holomorphic functions via the Cauchy-Riemann equations and differentiation under the integral sign via dominated convergence.

c) This can be found in the lecture notes. We leave the details to the reader.