1. a) This follows from Sh. 11, Ex. 3.
   b)
   c) The existence of z follows immediately from Minkowski's first theorem. If $|\sigma_1(z)| < 1$, then $|\sigma_i(z)| < 1/Y < 1$ for all $i = 2, \dots, r_1 + r_2$ implies that, letting $s_i = 1$ for $1 \leqslant i \leqslant r_1$ and $s_i = 2$ for $r_1 < i \leqslant r_2$, we have

$$|\det([\times z]_{K/\mathbb{Q}})| = |\mathrm{Nr}(z)| = \prod_{i=1}^{r_1+r_2} |\sigma_i(z)|^{s_i} < \left(\frac{1}{Y}\right)^{d-1} < 1,$$

   which is absurd since the left-hand side, since $[\times z]_{K/\mathbb{Q}}$ is invertible and its determinant equals the constant term of the characteristic polynomial of $[\times z]_{K/\mathbb{Q}}$, which was shown to have integer coefficients.
   d)
   e) If $\mathbb{Q}(z) \neq K$, then the preceding exercise implies that

$$|\{1 \leqslant i \leqslant d \colon \sigma_i(z) = \sigma_1(d)\}| \geqslant 2.$$

   It follows that there is $1 \leqslant i \leqslant r_2$ such that $\sigma_1(z) = \overline{\sigma_{r_1+i}(z)}$.
   Case 1: If $r_1 > 0$, then

$$\left|\overline{\sigma_{r_1+i}(z)}\right| = |\sigma_{r_1+i}(z)| < \frac{1}{Y} < 1 \leqslant |\sigma_1(z)|,$$

   which is absurd, and, hence, in this case it follows that $\mathbb{Q}(z) = K$.
   Case 2: If $r_1 = 0$, we know that $|\mathrm{Re}(\sigma_1(z_1))| < 1$ and $|\sigma_1(z)| \geqslant 1$, hence $\mathrm{Im}(\sigma_1(z)) \neq 0$. Hence $\sigma_1(z) \neq \overline{\sigma_1(z)}$ and, thus, there is $2 \leqslant j \leqslant r_2$ such that $\sigma_1(z) = \sigma_j(z)$. In particular

$$1 \leqslant |\sigma_1(z)| = |\sigma_j(z)| < \frac{1}{Y},$$

   which is absurd.
   f) The conclusion follows immediately.
2. We follow the hint. Let $C \subseteq \mathbb{R}^{r_1+r_2}$ compact. By Heine-Borel, there is $A > 0$ such that for any $v = (v_1, \dots, v_{r_1+r_2}) \in C$ we have

$$\forall 1 \leqslant i \leqslant r_1 + r_2 \quad -A \leqslant v_i \leqslant A.$$

   Suppose that $z \in \mathcal{O}_K^\times$ and suppose that $\mathrm{Log}_\infty(z) \in C$. Hence

$$\forall 1 \leqslant i \leqslant r_1 + r_2 \quad e^{-A} \leqslant |\sigma_i(z)|^{d_i} \leqslant e^A,$$

   where

$$d_i = \begin{cases} 1 & \text{if } 1 \leqslant i \leqslant r_1, \\ 2 & \text{else.} \end{cases}$$

   Let $P \in \mathbb{Z}[X]$ denote the characteristic polynomial of $[\times z]_{K/\mathbb{Q}}$. Then

$$P(X) = \prod_{i=1}^{d} \big(X - \sigma_i(z)\big).$$

Let $a_0, \dots, a_{d-1} \in \mathbb{Z}$ denote the coefficients of P, i.e.,

$$P(X) = X^d + a_{d-1}X^{d-1} + \cdots + a_0.$$

Then

$$\forall 0 \leqslant i < d \quad |a_i| \leqslant d! e^{Ad}.$$

The set

$$\{w \in \mathbb{R}^d \colon \forall 1 \leqslant i \leqslant d \, |w_i| \leqslant d! e^{Ad}\} \cap \mathbb{Z}^d$$

is finite, hence the set of $z \in \mathcal{O}_K^\times$ such that $\mathrm{Log}_\infty(z) \in C$ is finite. Now local compactness and the Hausdorff property for $\mathbb{R}^{r_1+r_2}$ imply that $\mathrm{Log}_\infty(\mathcal{O}_K^\times)$ is discrete. The kernel is the preimage of $\{0\}$ under $\mathrm{Log}_\infty|_{\mathcal{O}_K^\times}$. As singletons are compact in $\mathbb{R}^{r_1+r_2}$, the kernel is finite.

3. a) Recall that $G = \mathrm{Gal}(\mathbb{Q}(\zeta_8)/\mathbb{Q}) \cong (\mathbb{Z}/8\mathbb{Z})^\times$ is a group of cardinality 4, which is given by the group table

	1	3	5	7
1	1	3	5	7
3	3	1	7	5
5	5	7	1	3
7	7	5	3	1

In particular, $\mathrm{Gal}(\mathbb{Q}(\zeta_8)/\mathbb{Q})$ has exactly three subgroups of index 2 (and those are exactly the subgroups $H \lhd G$ such that $|G/H| = 2$). Note that $\mathbb{Q}(\zeta_8)$ certainly contains the Gaussian numbers $\mathbb{Q}(\mathrm{i}) = \mathbb{Q}(\zeta_8^2)$. Now let $M \subseteq \mathbb{Q}(\zeta_8)$ be any other quadratic number field. Then $L = \mathbb{Q}(\mathrm{i})M$ has degree four, since $[L\colon M] \geqslant 2$ and

$$4 = [\mathbb{Q}(\zeta_8)\colon \mathbb{Q}] = 2[\mathbb{Q}(\zeta_8)\colon L][L\colon M] \implies 1 \geqslant [\mathbb{Q}(\zeta_8)\colon L].$$

As of Sh. 10, Ex. 4, we know that

$$\mathrm{disc}\big(\mathbb{Q}(\zeta_8)\big) = \mathrm{disc}\big(\mathbb{Q}(\mathrm{i})\big)^2 \mathrm{disc}(M)^2 = 16\,\mathrm{disc}(M)^2.$$

As of Sh. 7, Ex. 2, we know that

$$\mathrm{disc}\big(\mathbb{Q}(\zeta_8)\big) = 2^8,$$

hence $|\mathrm{disc}(M)| = 2^3 = 8$. It follows that $M = \mathbb{Q}(\sqrt{d})$ for $d \in \{\pm 2\}$ and, since $\mathrm{i} \in \mathbb{Q}(\zeta_8)$, we have

$$\mathbb{Q}(\mathrm{i}), \mathbb{Q}\big(\sqrt{2}\big), \mathbb{Q}\big(\sqrt{-2}\big) \subseteq \mathbb{Q}(\zeta_8).$$

b) The extension $\mathbb{Q}(\zeta_p)|\mathbb{Q}$ is Galois with abelian Galois group A of order $p-1$. Recall that the multiplicative subgroup of a finite field is cyclic. Hence A contains a unique subgroup B of index $\frac{p-1}{2}$. Indeed, if $H, M < A$ are distinct subgroups of cardinality $\frac{p-1}{2}$, then $HM < A$ is a subgroup strictly containing H and, therefore, $HM = A$. In particular, HM contains an element of order $p-1$. Any such element is of the form hm with $h \in H$ and $m \in M$. The order of hm is $\mathrm{lcm}(a,b)$, where a and b denote the orders of h and m respectively. By assumption, $\mathrm{lcm}(a,b) | \frac{p-1}{2}$, which contradicts the assumption that H, M were distinct.
Let $M \subseteq \mathbb{Q}(\zeta_p)$ be the subfield given by $\mathrm{Gal}(M/\mathbb{Q}) \cong A/B$. Since $[A\colon B] = 2$, we find that M is quadratic.
Suppose that q is a prime that divides the discriminant of M, then q ramifies in M and, therefore, also ramifies in $\mathbb{Q}(\zeta_p)$. Since the only prime that ramifies in $\mathbb{Q}(\zeta_p)$ is p, it follows that $q = p$.

In particular, we have $M = \mathbb{Q}(\sqrt{\varepsilon p})$, where $\varepsilon \in \{\pm 1\}$. Note that the discriminant d of M has to satisfy $d \equiv 0, 1 \mod 4$ and, since 2 is unramified in $\mathbb{Q}(\zeta_p)$, we know that $d \equiv 1 \mod 4$. Hence

$$\varepsilon = \begin{cases} 1 & \text{if } p \equiv 1 \mod 4, \\ -1 & \text{if } p \equiv 3 \mod 4. \end{cases}$$

More concisely, letting $p^* = (-1)^{\frac{p-1}{2}} p$, we have shown that $\mathbb{Q}(\sqrt{p^*})$ is the unique quadratic subfield of $\mathbb{Q}(\zeta_p)$.

c) Using Sh. 10, Ex. 5 and the preceding exercise, we obtain that for any collection $p_1, \ldots, p_\ell$ of pairwise distinct odd primes, letting

$$\varepsilon = \prod_{i=1}^{\ell} (-1)^{\frac{p_i - 1}{2}},$$

we have

$$\mathbb{Q}(\sqrt{\varepsilon p_1 \cdots p_\ell}) \subseteq \mathbb{Q}(\sqrt{p_1^*}) \cdots \mathbb{Q}(\sqrt{p_\ell^*}) \subseteq \mathbb{Q}(\zeta_{p_1}) \cdots \mathbb{Q}(\zeta_{p_\ell}) = \mathbb{Q}(\zeta_{p_1 \cdots p_\ell}).$$

The general case now becomes a case distinction.

Case 1: If $D \equiv 1 \mod 4$ and $\Delta = D$, then D is a product of pairwise distinct odd primes and the number of prime factors $p | D$ satisfying $p \equiv 3 \mod 4$ is even. Hence, in the notation used above, $\varepsilon = 1$ and therefore we have shown by the preceding argument that $K = \mathbb{Q}(\sqrt{\Delta}) \subseteq \mathbb{Q}(\zeta_D)$.

Case 2: If $D \equiv 1 \mod 4$ and $\Delta = -D$, then $\Delta \equiv 3 \mod 4$, which is not possible, as we have shown in Sh. 3, Ex. 5 that all discriminants of (quadratic) number fields are $0, 1 \mod 4$.

Case 3: If $D \equiv 0 \mod 4$, $D/4 \equiv 2 \mod 4$, and $D = \Delta$, then $\Delta = 8 p_1 \cdots p_\ell$ for p_i pairwise distinct odd primes such that the number of prime factors $p_i \equiv 3 \mod 4$ is even. Again, we obtain that in the notation used above, $\varepsilon = 1$ and therefore

$$\mathbb{Q}(\sqrt{\Delta}) \subseteq \mathbb{Q}(\sqrt{2}) \mathbb{Q}(\sqrt{p_1 \cdots p_\ell}) \subseteq \mathbb{Q}(\zeta_8) \mathbb{Q}(\zeta_{p_1 \cdots p_\ell}) = \mathbb{Q}(\zeta_{8 p_1 \cdots p_\ell}) = \mathbb{Q}(\zeta_D).$$

Case 4: If $D \equiv 0 \mod 4$, $D/4 \equiv 2 \mod 4$, and $D = -\Delta$, then $\Delta = -8 p_1 \cdots p_\ell$ for p_i pairwise distinct odd primes such that the number of prime factors $p_i \equiv 3 \mod 4$ is even. Hence, by the same reasoning as above

$$\mathbb{Q}(\sqrt{\Delta}) \subseteq \mathbb{Q}(\sqrt{-2}) \mathbb{Q}(\sqrt{p_1 \cdots p_\ell}) \subseteq \mathbb{Q}(\zeta_8) \mathbb{Q}(\zeta_{p_1 \cdots p_\ell}) = \mathbb{Q}(\zeta_{8 p_1 \cdots p_\ell}) = \mathbb{Q}(\zeta_D).$$

Case 5: If $D \equiv 0 \mod 4$ and $D/4 \equiv 1 \mod 4$, then $D = -\Delta$ and $\Delta = -p_1 cdots p_\ell$ for p_i pairwise distinct odd primes such that the number of prime factors $p_i \equiv 3 \mod 4$ is even. Hence $\varepsilon = 1$ and therefore

$$\mathbb{Q}(\sqrt{\Delta}) \subseteq \mathbb{Q}(\mathrm{i}) \mathbb{Q}(\sqrt{p_1 \cdots p_\ell}) \subseteq \mathbb{Q}(\zeta_4) \mathbb{Q}(\zeta_{p_1 \cdots p_\ell}) = \mathbb{Q}(\zeta_D).$$

Case 6: If $D \equiv 0 \mod 4$ and $D/4 \equiv 3 \mod 4$, then $D = \Delta$ and hence $\Delta = p_1 \cdots p_\ell$ for pairwise distinct odd primes p_i and an odd number of prime factors satisfying $p_i \equiv 3 \mod 4$. Hence $\varepsilon = -1$ and therefore

$$\mathbb{Q}(\sqrt{\Delta}) \subseteq \mathbb{Q}(\mathrm{i}) \mathbb{Q}(\sqrt{-p_1 \cdots p_\ell}) \subseteq \mathbb{Q}(\zeta_4) \mathbb{Q}(\zeta_{p_1 \cdots p_\ell}) = \mathbb{Q}(\zeta_D).$$

d) Let $K = \mathbb{Q}(\sqrt{d})$ with $d \in \mathbb{Z}$ squarefree. Remember that the isomorphism

$$(\mathbb{Z}/D\mathbb{Z})^\times \cong \mathrm{Gal}(\mathbb{Q}(\zeta_D)/\mathbb{Q})$$

is given by the map which sends $a \in (\mathbb{Z}/D\mathbb{Z})^\times$ to the field automorphism defined by $\zeta_D \mapsto {\zeta_D}^a$.

So, consider $-1 \in (\mathbb{Z}/D\mathbb{Z})^\times$. In $\mathrm{Gal}(\mathbb{Q}(\zeta_D)/\mathbb{Q})$, this corresponds to the automorphism $\phi\colon \mathbb{Q}(\zeta_D) \to \mathbb{Q}(\zeta_D)$ given by $\phi(\zeta_D) = {\zeta_D}^{-1} = \overline{\zeta_D}$. If K is a real quadratic field, i.e., if $\Delta > 0$, then the restriction of ϕ to K is simply the identity map, and thus $\chi_K(-1) = 1$. On the other hand, if K is an imaginary quadratic field, i.e., if $\Delta < 0$, the restriction of ϕ to K is complex conjugation, which is not the identity map, and thus we must have $\chi_K(-1) = -1$.

e) Let $\mathfrak{p} \lhd \mathcal{O}_K$ be a prime ideal over p, and let $\mathfrak{P}$ be a prime ideal of $\mathcal{O}_{\mathbb{Q}(\zeta_D)}$ over $\mathfrak{p}$. Then there is a natural injection $k_\mathfrak{p} \to k_\mathfrak{P}$ between the residue fields of $\mathfrak{p}$ and $\mathfrak{P}$, which induces a surjective group homomorphism

$$\gamma\colon \mathrm{Gal}(k_\mathfrak{P}/\mathbb{F}_p) \to \mathrm{Gal}(k_\mathfrak{p}/\mathbb{F}_p).$$

Note that under this last homomorphism, the Frobenius map $\mathrm{Frob}_{p|k_\mathfrak{P}}$ is sent to the Frobenius map $\mathrm{Frob}_{p|k_\mathfrak{p}}$.

The surjective group homomorphism $\mathrm{Gal}(\mathbb{Q}(\zeta_D)/\mathbb{Q}) \to \mathrm{Gal}(K/\mathbb{Q})$ induces a group homomorphism $\delta\colon D_\mathfrak{P} \to D_\mathfrak{p}$ between the decomposition groups of $\mathfrak{P}$ and $\mathfrak{p}$. Since, by assumption, p is not ramified, we have

$$D_\mathfrak{P} \cong \mathrm{Gal}(k_\mathfrak{P}/\mathbb{F}_p) \quad \text{and} \quad D_\mathfrak{p} \cong \mathrm{Gal}(k_\mathfrak{p}/\mathbb{F}_p),$$

and under these isomorphisms, the Frobenius element $(p, \mathbb{Q}(\zeta_D)/\mathbb{Q})$ corresponds to $\mathrm{Frob}_{p|k_\mathfrak{P}}$, while $(p, K/\mathbb{Q})$ corresponds to $\mathrm{Frob}_{p|k_{\mathfrak{p}i}}$.

Furthermore, the homomorphism δ between $D_\mathfrak{P}$ and $D_\mathfrak{p}$ corresponds to the homomorphism γ between the groups $\mathrm{Gal}(k_\mathfrak{P}/\mathbb{F}_p)$ and $\mathrm{Gal}(k_\mathfrak{p}/\mathbb{F}_p)$. In other words, we have a commutative diagram:

$$\begin{array}{ccc} D_\mathfrak{P} & \xrightarrow{\quad\delta\quad} & D_\mathfrak{p} \\ \big\downarrow \wr & & \big\downarrow \wr \\ \mathrm{Gal}(k_\mathfrak{P}/\mathbb{F}_p) & \xrightarrow{\gamma} & \mathrm{Gal}(k_\mathfrak{p}/\mathbb{F}_p) \end{array}$$

From these observations it is now clear that under the homomorphism δ the element $(p, \mathbb{Q}(\zeta_D)/\mathbb{Q})$ is indeed sent to $(p, K/\mathbb{Q})$, as we wanted to show.

f) Note that, under the identification

$$(\mathbb{Z}/D\mathbb{Z})^\times \cong \mathrm{Gal}(\mathbb{Q}(\zeta_D)/\mathbb{Q}),$$

the element $p \in (\mathbb{Z}/D\mathbb{Z})^\times$ corresponds to the Frobenius element $(p, \mathbb{Q}(\zeta_D)/\mathbb{Q})$. As we have just shown above, $\chi(p)$ then corresponds to the Frobenius element $(p, K/\mathbb{Q})$. The latter is trivial if and only if $[k_\mathfrak{p}\colon \mathbb{F}_p] = 1$, which is the case if and only if p splits in K. On the other hand, $(p, K/\mathbb{Q})$ is non-trivial if and only if $[k_\mathfrak{p}\colon \mathbb{F}_p] = 2$, which happens if and only if p is inert in K.

4. a) Let $\zeta \in \mathbb{Q}(\sqrt{d})$ a root of unity. As ζ is real, we know that $\zeta \in \{\pm 1\}$. As $\mathrm{rank}(\mathcal{O}_K^\times) = 1$,

$$\mathrm{Log}_\infty(\mathcal{O}_K^\times) \subseteq \{(x, y) \in \mathbb{R}^2\colon y = -x\}$$

has exactly two generators: There is $\mathbf{v}$ in $\mathrm{Log}_\infty(\mathcal{O}_K^\times)$ such that any generator $\mathbf{w}$ of $\mathrm{Log}_\infty(\mathcal{O}_K^\times)$ satisfies $\mathbf{w} \in \{\pm 1\}$.

Let $\mathbf{v} = (x, -x)$ be a generator of $\mathrm{Log}_\infty(\mathcal{O}_K^\times)$ and let $\varepsilon \in \mathcal{O}_K^\times$ be a preimage of $\mathbf{v}$. Then ε is well-defined up to multiplication by a root of unity in $\mathbb{Q}(\sqrt{d})$. Hence there is exactly one preimage ε satisfying $\varepsilon > 0$. By the choice of enumeration of embeddings, we have $\varepsilon > 1$ if and only if $x > 0$, and thus uniqueness follows.

b) Recall that $\mathcal{O}_K = \mathbb{Z}[\frac{D+\sqrt{D}}{2}]$, where D is the discriminant of $\mathbb{Q}(\sqrt{d})$, i.e., $D = d$ if $d \equiv 1 \mod 4$ and $D = 4d$ if $d \equiv 2, 3 \mod 4$. In particular, we find that

$$\varepsilon_d = \frac{a + b\sqrt{d}}{2}$$

for some integers $a, b \in \mathbb{Z}$.
Assume that $ab < 0$, then

$$\frac{a + b\sqrt{d}}{2} = \frac{2\mathrm{Nr}_{K/\mathbb{Q}}(\varepsilon_d)}{a - b\sqrt{d}}.$$

But $|a - b\sqrt{d}| = |a| + |b|\sqrt{d} > 2$ gives a contradiction.
Note that $ab \neq 0$:

$$\begin{aligned} a = 0 &\implies 1 = |\mathrm{Nr}_{K/\mathbb{Q}}(\varepsilon_d)| = \frac{b^2}{4}d \\ &\implies d = 1 \text{ since } d \text{ is square-free.} \\ b = 0 &\implies \varepsilon_d \in \mathcal{O}_K \cap \mathbb{Q} = \mathbb{Z} \implies \varepsilon_d \in \{\pm 1\}. \end{aligned}$$

It follows that $a, b > 0$ since they have the same sign and $\varepsilon_d > 0$ by assumption.

c) By the preceding argument, we know that

$$U_{\text{fun}} \cap]1, X] \subseteq \left\{ \frac{m + n\sqrt{d}}{2} : m \in [1, 2X], n^2 d \in [1, 4X^2] \right\},$$

and the right-hand side is clearly finite.

d) Note that $1 < u < X$ implies that there is some $k \in \mathbb{N}$ such that $u = \varepsilon_d^k$. Let $\alpha_k, \beta_k \in \mathbb{Q}$ be given by

$$\varepsilon_d^k = \alpha_k + \beta_k\sqrt{d} \quad (k \in \mathbb{N}).$$

For all $k \in \mathbb{N}$

$$\begin{aligned} \alpha_{k+1} &= \alpha_1\alpha_k + d\beta_1\beta_k, \\ \beta_k &= \alpha_1\beta_k + \alpha_k\beta_1. \end{aligned}$$

Hence $1 < u = a + b\sqrt{d} < X$ implies, in particular, that $a, b > 0$ (since $\alpha_1, \beta_1 > 0$ by part b).
We treat the cases $a = 1/2$ and $a = 1$ separately. If $a = 1/2$, then $\pm 1 = a^2 - db^2$ with b a half-integer implies that $u = \frac{1}{2} + \frac{1}{2}\sqrt{5}$. If $a = 1$, then $\pm 1 = a^2 - db^2$ implies that implies $u = 1$ or $u = 1 + \sqrt{2}$. So we can assume $a > 1$. Again, using $a^2 - b^2 d = \pm 1$, we obtain that $b^2 d = a^2 \pm 1$ and hence, since b is positive, $b\sqrt{d} = \sqrt{a^2 \pm 1}$. Now the upper bound follows from $u = a + \sqrt{a^2 \pm 1}$:

$$\begin{aligned} u < X &\iff a + \sqrt{a^2 \pm 1} < X \\ &\iff a^2 \pm 1 < X^2 + a^2 - 2aX \\ &\iff a < (X^2 \pm 1)/2X \end{aligned}$$

e) We can again assume that $a > 1$. The number $u = a + b\sqrt{d}$ has norm σ if and only if $a^2 - \sigma = b^2 d$. Write $a = m/2$, then we have

$$m^2 - 4\sigma = 4b^2 d.$$

Note that $m^2 - 4\sigma > 0$ since $m \geqslant 3$. Using the fundamental theorem of arithmetic, it is clear that $m^2 - 4\sigma$ admits a unique factorization of the form dn^2, where $d \in \mathbb{N}$ is square-free and $n \in \mathbb{N}$.

f) Using parts (d) and (e), we have to show that

$$2|]1, (X^2 \pm 1)/2X] \cap \tfrac{1}{2}\mathbb{Z}| = 2X + O(1),$$

where the leading 2 on the leaft-hand side comes from the fact that there are two possible signs σ to choose from. Since $(X^2 \pm 1)/2X = X/2 + O(1)$, the interval $]1, (X^2 \pm 1)/2X]$ contains $X + O(1)$-many half integers. Thus the claim follows.

g) An element $u \in U_{\mathrm{all}}$ lies in $]1, X]$ if and only if there is a fundamental unit $\varepsilon \in U_{\mathrm{k}}$ and some $k \in \mathbb{N}$ such that $u = \varepsilon^k$ and $\varepsilon \in]1, X^{1/k}]$. Thus the claim is immediate.
It remains to show that, for given $X > 1$, there is $k \in \mathbb{N}$ such that $f(X^{1/k}) = 0$. From this, since f is non-decreasing, it follows that the sum in question is indeed finite. One checks that for every fundamental unit $u \in U_{\mathrm{fun}}$ we have $u \geqslant 3/2$. Hence, whenever $k \geqslant \lceil \log X/(\log 3 - \log 2) \rceil$, we have $f(X^{1/k}) = 0$.

h) We start with a claim. Let $g, h \colon \mathbb{N} \to \mathbb{C}$ such that h has finite support and g is bounded, then the function $F \colon \mathbb{N}^2 \to \mathbb{C}$ given by $F(k, \ell) = g(k)h(k\ell)$ is in $\mathrm{L}^1(\mathbb{N}^2)$ and

$$\sum_{k \in \mathbb{N}} \sum_{\ell \in \mathbb{N}} F(k, \ell) = \sum_{n \in \mathbb{N}} h(n) \sum_{d \mid n} g(d).$$

In order to prove summability, suppose that $M > 0$ is such that $\|g\|_\infty \leqslant M$ and let $n_0 \in \mathbb{N}$ such that

$$n > n_0 \implies h(n) = 0.$$

Then

$$\sum_{k \in \mathbb{N}} \sum_{\ell \in \mathbb{N}} |F(k, \ell)| \leqslant M \sum_{k \in \mathbb{N}} \sum_{\ell \in \mathbb{N}} |h(k\ell)| = M \sum_{\ell, k \leqslant n_0} |h(k\ell)| < \infty,$$

since the last sum runs over a finite set. Thus Tonelli's theorem implies that $F \in \mathrm{L}^1(\mathbb{N}^2)$. In particular, the sum is independent of the order of summation, i.e., we can sum over shells. More precisely, by definition of the Lebesgue integral, we have that

$$\sum_{k, \ell \in \mathbb{N}} F(k, \ell) = \lim_{N \to \infty} \sum_{k\ell \leqslant N} F(k, \ell) = \sum_{n \in \mathbb{N}} \sum_{k\ell = n} F(k, \ell).$$

Hence

$$\sum_{k, \ell \in \mathbb{N}} F(k, \ell) = \sum_{n \in \mathbb{N}} \sum_{k\ell = n} g\left(\frac{n}{\ell}\right) h(n) = \sum_{n \in \mathbb{N}} h(n) \sum_{d \mid n} g(d).$$

We apply this with $g(k) = \mu(k)$ and $h(\ell) = f(X^{1/\ell})$ to obtain

$$\begin{aligned} \sum_{k=1}^{\infty} \mu(k) a(X^{1/k}) &= \sum_{k=1}^{\infty} \mu(k) \sum_{\ell=1}^{\infty} f(X^{\frac{1}{k\ell}}) \\ &= \sum_{n=1}^{\infty} f(X^{1/n}) \sum_{d \mid n} \mu(d) = f(X). \end{aligned}$$

i) Recall that there is some $M \in \mathbb{N}$ such that

$$k \geqslant M \log X \implies f(X^{1/k}) = 0.$$

Hence, using part (g) and the fact that f is monotonic, we obtain that

$$|f(X) - a(X)| \leqslant M(\log X) f(X^{1/2}).$$

Since $f(X^{1/2}) \leqslant a(X^{1/2})$, we thus obtain that

$$f(X) = a(X) + O(X^{1/2}(\log X)) = 2X + o(X)$$

as desired. In particular,

$$\lim_{X \to \infty} \frac{|U_{\text{fun}} \cap]1, X]|}{X} = \lim_{X \to \infty} \frac{a(X)}{X} = 2.$$