1. a)
   b)
   c) Let $\varepsilon > 0$. Let $\mathcal{P}_\varepsilon$ be the set defined as
$$\mathcal{P}_\varepsilon := \{p \text{ prime} : p \leqslant e^{\frac{d}{\varepsilon}}\}.$$
By the definition of $\mathcal{P}_\varepsilon$ and by the well-known inequality $\ell + 1 \leqslant e^\ell$, which holds for all $\ell \in \mathbb{N}$, we have that
$$\frac{(\ell+1)^d}{p^{\varepsilon\ell}} \leqslant 1 \qquad \text{for all} \quad p \notin \mathcal{P}_\varepsilon \quad \text{and} \quad \ell \in \mathbb{N}.$$
Next, we define the positive real number M_ε as follows,
$$M_\varepsilon := \max_{\lambda\in[0,\infty)} \frac{(\lambda+1)^d}{2^{\varepsilon\lambda}}.$$
Now, given a positive integer n with prime decomposition
$$n = {p_1}^{\ell_1} \cdots {p_r}^{\ell_r},$$
it follows from the results proven in 1a and 1b that
$$\frac{r_K(n)}{n^\varepsilon} \leqslant \prod_{i=1}^{r} \frac{(\ell_i+1)^d}{{p_i}^{\varepsilon\ell_i}} \leqslant \prod_{\substack{1\leqslant i\leqslant r\\ p_i\in\mathcal{P}_\varepsilon}} \frac{(\ell_i+1)^d}{{p_i}^{\varepsilon\ell_i}} \leqslant \prod_{\substack{1\leqslant i\leqslant r\\ p_i\in\mathcal{P}_\varepsilon}} \frac{(\ell_i+1)^d}{2^{\varepsilon\ell_i}} \leqslant {M_\varepsilon}^{|\mathcal{P}_\varepsilon|}.$$
Thus
$$r_K(n) \leqslant C_\varepsilon n^\varepsilon,$$
where we have set $C_\varepsilon := {M_\varepsilon}^{|\mathcal{P}_\varepsilon|}$.
   d)
2. a) We first prove the claim for the special case where $\mathfrak{a} \lhd \mathcal{O}_K$. As of Sh. 2 Ex. 2a it suffices to show that
$$\operatorname{covol}(\sigma_\infty(\mathcal{O}_K)) = 2^{-r_2}|\operatorname{disc}(K)|^{\frac{1}{2}}.$$
Let $(\alpha_1, \ldots, \alpha_d) \in \mathcal{O}_K^d$ be a $\mathbb{Z}$-basis of $\mathcal{O}_K$. Then
$$\big(\sigma_\infty(\alpha_1), \ldots, \sigma_\infty(\alpha_d)\big) \in K_\infty$$
is a $\mathbb{Z}$-basis for $\sigma_\infty(\mathcal{O}_K)$. We recall that for $r_1 < i \leqslant r_2$ and $z \in \mathcal{O}_K$ we have
$$\underbrace{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}_{=A} \begin{pmatrix} \operatorname{Re}\sigma_i(z) \\ \operatorname{Im}\sigma_i(z) \end{pmatrix} = \begin{pmatrix} \sigma_i(z) \\ \overline{\sigma_i}(z) \end{pmatrix}$$
and therefore
$$\begin{aligned} \operatorname{covol}\big(\sigma_\infty(\mathcal{O}_K)\big) &= \det\big(\sigma_\infty(\alpha_1), \ldots, \sigma_\infty(\alpha_d)\big) \\ &= (\det A)^{-r_2} \underbrace{\det \begin{pmatrix} \sigma_1(\alpha_1) & \cdots & \sigma_1(\alpha_d) \\ \vdots & \ddots & \vdots \\ \sigma_d(\alpha_1) & \cdots & \sigma_d(\alpha_d) \end{pmatrix}}_{=\Delta}. \end{aligned}$$

We have seen that $\Delta^2 = \mathrm{disc}(K)$. As $\det A = -2$, the claim follows.
Now suppose that $\mathfrak{f} \lhd K$ is a non-zero fractional ideal. We first define a $\mathbb{Q}$-linear embedding $\iota\colon K \to \mathrm{Mat}_d(\mathbb{R})$ as follows. Fix a $\mathbb{Z}$-basis $(\omega_1, \dots, \omega_d) \in \mathcal{O}_K^d$ of $\mathcal{O}_K$ and recall that $\big(\sigma_\infty(\omega_1), \dots, \sigma_\infty(\omega_d)\big) \in (\mathbb{R}^d)^d$ is a basis of $\mathbb{R}^d$. Given $b \in K$, we define $\iota(b) \in \mathrm{Mat}_d(\mathbb{R})$ by the unique linear extension of

$$\forall 1 \leqslant i \leqslant d \quad \sigma_\infty(\omega_i)\iota(b) = \sigma_\infty(\omega_i b).$$

Note that the diagram

$$\begin{tikzcd} K \arrow[r, "\sigma_\infty"] \arrow[d, "{[\times b]_{K/\mathbb{Q}}}"'] & \mathbb{R}^d \arrow[d, "\iota(b)"] \\ K \arrow[r, "\sigma_\infty"'] & \mathbb{R}^d \end{tikzcd}$$

commutes. Hence density of $\sigma_\infty(K) \subseteq \mathbb{R}^d$ implies that $\det\big(\iota(b)\big) = \det([\times b]_{K/\mathbb{Q}})$.
Now suppose that $b \in \mathcal{O}_K$ and $\mathfrak{a} \lhd \mathcal{O}_K$ satisfy that $\mathfrak{f} = b^{-1}.\mathfrak{a} = [\times b]_{K/\mathbb{Q}}(\mathfrak{a})$. Then

$$\begin{aligned} \mathrm{covol}\big(\sigma_\infty(\mathfrak{f})\big) &= \mathrm{covol}\big(\sigma_\infty([\times b^{-1}]_{K/\mathbb{Q}}(\mathfrak{a}))\big) \\ &= \mathrm{covol}\big(\sigma_\infty(\mathfrak{a})\iota(b)^{-1}\big) \\ &= \big|\det\big(\iota(b)\big)\big|^{-1}\mathrm{covol}\big(\sigma_\infty(\mathfrak{a})\big) \\ &= |\mathrm{Nr}_{K/\mathbb{Q}}(b)|^{-1}2^{-r_2}|\mathrm{disc}(\mathcal{O}_K)|^{\frac{1}{2}}\mathrm{Nr}(\mathfrak{a}) \\ &= 2^{-r_2}|\mathrm{disc}(\mathcal{O}_K)|^{\frac{1}{2}}\mathrm{Nr}(\mathfrak{f}). \end{aligned}$$

b) Let $(\alpha_1, \dots, \alpha_d) \in K^d$ a $\mathbb{Q}$-basis of K. Then $(\alpha_1 \otimes 1, \dots, \alpha_d \otimes 1) \in (K \otimes_{\mathbb{Q}} \mathbb{R})^d$ is an $\mathbb{R}$-basis of $K \otimes_{\mathbb{Q}} \mathbb{R}$. We define a linear map $\Phi\colon K \otimes_{\mathbb{Q}} \mathbb{R} \to \mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2}$ by linearly extending

$$\Phi(\alpha_i \otimes 1) = \sigma_\infty(\alpha_i) \quad (1 \leqslant i \leqslant d).$$

The verification that Φ defines a homomorphism of $\mathbb{R}$-algebras is purely formal. As

$$\dim_{\mathbb{R}}(K \otimes \mathbb{R}) = \dim_{\mathbb{Q}} K = d = \dim_{\mathbb{R}}(\mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2}),$$

it suffices to prove that Φ is injective. This follows from

$$\big(\Phi(\alpha_1 \otimes 1), \dots, \Phi(\alpha_d \otimes 1)\big) = (\sigma_\infty\alpha_1, \dots \sigma_\infty\alpha_d)$$

being linearly independent over $\mathbb{R}$ as was proven in class.

3. One calculates

$$a_2 = \left(\frac{4}{2}\frac{\pi}{4}\right)^2 = \frac{\pi^2}{4} > 2$$

and

$$\frac{a_{d+1}}{a_d} = \left(\frac{(d+1)^{d+1}}{d^d}\frac{1}{d+1}\left(\frac{\pi}{4}\right)^{\frac{1}{2}}\right)^2 = \frac{\pi}{4}\left(1 + \frac{1}{d}\right)^{2d}.$$

Now we note that

$$\left(1 + \frac{1}{d}\right)^{2d} = 1 + 2d\frac{1}{d} + \sum_{k=2}^{2d}\binom{2d}{k}d^{-k} \geqslant 3$$

and conclude that

$$\frac{a_{d+1}}{a_d} \geqslant \frac{3\pi}{4} > 1.$$

As of Minkowski's bound, we can find a non-zero ideal $\mathfrak{a} \lhd \mathcal{O}_K$ such that

$$1 \leqslant \mathrm{Nr}(\mathfrak{a})^2 \leqslant \left(\frac{d!}{d^d} \left(\frac{4}{\pi} \right)^{r_2} \right)^2 |\mathrm{disc}(K)|$$

and hence

$$|\mathrm{disc}(K)| \geqslant \left(\frac{d^d}{d!} \right)^2 \left(\frac{\pi}{4} \right)^{2r_2} \geqslant \left(\frac{d^d}{d!} \right)^2 \left(\frac{\pi}{4} \right)^{d} = a_d \geqslant 2.$$

4. Given a number field $K/\mathbb{Q}$, we let $\Delta_K = \mathrm{disc}(\mathcal{O}_K)$. Given $d, n \geqslant 2$ integers, define $\gamma_d(n)$ by

$$n^{\gamma_d(n)} = \frac{d!}{d^d} \left(\frac{4}{\pi} \right)^{r_2}.$$

Hence, if K is a number field of degree $d \geqslant 2$, since $|\Delta_K| \geqslant 2$, we obtain

$$\frac{d!}{d^d} \left(\frac{4}{\pi} \right)^{r_2} |\Delta_K|^{\frac{1}{2}} = |\Delta_K|^{\frac{1}{2} + \gamma_d(|\Delta_K|)}.$$

As of Ex. 1d and Minkowski's bound, for every $\varepsilon > 0$ there is $C_\varepsilon > 0$ such that

$$|\mathrm{Cl}(K)| \leqslant \left| \left\{ \mathfrak{a} \lhd \mathcal{O}_K \colon \mathrm{Nr}(\mathfrak{a}) \leqslant |\Delta_K|^{\frac{1}{2} + \gamma_d(|\Delta_K|)} \right\} \right| \leqslant C_\varepsilon |\Delta_K|^{\frac{1}{2} + \frac{\varepsilon}{2} + 2\gamma_d(|\Delta_K|)}.$$

Note that $\lim_{n \to \infty} \gamma_d(n) = 0$. Hence the claim follows after enlarging C_ε to account for the finitely many n satisfying $n^{\frac{\varepsilon}{2} + 2\gamma_d(n)} \geqslant n^\varepsilon$.

5.

6. As proven in the preceding exercise, $\mathrm{Cl}(K)$ is generated by prime ideals lying above primes satisfying the bound

$$1 \leqslant a \leqslant \frac{4}{\pi} \cdot \frac{2!}{2^2} \cdot \sqrt{19} < 3.$$

In order to determine $\mathrm{Cl}(K)$, we thus need to find the factorization of (2).

Note that we have

$$\mathcal{O}_K = \mathbb{Z}[\theta] \quad \text{with} \quad \theta = \frac{1 + \sqrt{-19}}{2},$$

with the minimal polynomial of θ given by

$$P(X) = X^2 - X + 5.$$

Now, observing that

$$P(X) \equiv X^2 - X + 1$$

and since $P(X)$ is irreducible $\mod 2$ using Dedekind recipe, we see that (2) is in fact prime. Hence every ideal in $\mathcal{O}_K$ is equivalent to a principal ideal. In other words, $\mathrm{Cl}(\mathcal{O}_K)$ is trivial.

7.

8.

```
# PRE:  Integer D
# POST: If D is not a perfect square, the class number of
#       the quadratic field QQ(sqrt(D)). Otherwise 1.
def ClassNumber(D):
    if is_square(D):
        return 1
    else:
        z = sqrt(D)
        K.<b> = QQ[z]
        return K.class_number()

```

```python
# PRE:  Positive integers T and n
# POST: The proportion of number fields with class number
#       n among quadratic fields of discriminant between 2
#       and T.
def ProportionClassNumber(T,n):
    N = 0
    R = 0
    for D in range(T):
        if is_fundamental_discriminant(D):
            N += 1
            h = ClassNumber(D)
            if h == n:
                R += 1
    return R/N

# PRE:  Positive integers T and n
# POST: The proportion of number fields with class number
#       n among quadratic fields of prime discriminant
#       between 2 and T.
def ProportionClassNumberPrime(T,n):
    N = 0
    R = 0
    for D in range(T):
        if is_fundamental_discriminant(D) and is_prime(D):
            N += 1
            h = ClassNumber(D)
            if h == n:
                R += 1
    return R/N
```