The class number formula

Theorem

Let $K/\mathbb{Q}$ a #-field, $[K:\mathbb{Q}]=d=r_1+2r_2$. Then

$$\left|\{\mathfrak{a}\triangleleft\mathcal{O}_K : \mathrm{Nr}(\mathfrak{a})\leq X\}\right| = C_K h(K)\,\mathrm{reg}(K)\,X + O_K\left(X^{1-\frac{1}{d}}\right)$$

We have seen last time that

$$\left|\{\underline{\mathfrak{a}}\triangleleft\mathcal{O}_K : \mathrm{Nr}(\underline{\mathfrak{a}})\leq X\}\right| = \sum_{[\mathfrak{a}]=g\in Cl(K)} \left| \underbrace{\{a\in\underline{\mathfrak{a}}-\{0\} : |\mathrm{Nr}(a)|\leq X\,\mathrm{Nr}(\underline{\mathfrak{a}})\}}_{\substack{\cong \#\text{ of lattice points} \\ \text{in a (non-convex) domain} \\ \rightsquigarrow \text{ counting problem.}}} \Big/ \mathcal{O}_K^\times \right|$$

sum of size $h(K)$

Some notation:

Recall that $K_\infty = \mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2} \cong K \otimes_{\mathbb{Q}} \mathbb{R}$. Then $K_\infty^\times = \{(x_1, \dots, x_{r_1}, z_1, \dots, z_{r_2}) : x_1 \cdots x_{r_1} z_1 \cdots z_{r_2} \neq 0\}$.

Moreover, $\mathcal{O}_K^\times \curvearrowright K_\infty^\times$ by $\varepsilon \cdot v := (\sigma_1(\varepsilon)x_1, \dots, \sigma_{r_1}(\varepsilon)x_{r_1}, \sigma_{r_1+1}(\varepsilon)z_1, \dots, \sigma_{r_1+r_2}(\varepsilon)z_{r_2})$

Definition:

Let $Nr : K_\infty^\times \longrightarrow \mathbb{R}_{>0}$, $Nr(v) = \prod_{i=1}^{r_1+r_2} |v_i|^{d_i}$

Then
- $Nr(\sigma_\infty(z)) = |Nr_{K|\mathbb{Q}}(z)|$ for all $z \in K$.
- Nr is $\mathcal{O}_K^\times$-invariant: $Nr(\sigma_\infty(\varepsilon)v) = Nr(v)$
- Homogeneous of degree $d = [K:\mathbb{Q}]$ (for $\lambda \in \mathbb{R}_{>0}$)
- $K_{\infty,X}^\times := \{v \in K_\infty^\times : Nr(v) = X\} = X^{\frac{1}{d}} K_{\infty,1}^\times$ and

  $K_{\infty,\leq X}^\times := \bigcup_{0<X'\leq X} K_{\infty,X'}^\times = X^{\frac{1}{d}} K_{\infty,\leq 1}^\times$.

Recall: We want to determine

$$\Big|\underbrace{\{a\in\underline{\mathfrak{a}}-\{0\} : |\mathrm{Nr}_{K|\mathbb{Q}}(a)|\le X\,\mathrm{Nr}(\underline{\mathfrak{a}})\}}_{\underline{\mathfrak{a}}_X}/\mathcal{O}_K^\times\Big| \qquad (\underline{\mathfrak{a}}\in\mathcal{J}).$$

Note that

$$\underline{\mathfrak{a}}_X \longleftrightarrow \{v\in\sigma_\infty(\underline{\mathfrak{a}})-\{0\} : \mathrm{Nr}(v)\le X\,\mathrm{Nr}(\underline{\mathfrak{a}})\}$$
$$=\sigma_\infty(\underline{\mathfrak{a}})\cap\left(X\,\mathrm{Nr}(\underline{\mathfrak{a}})\right)^{\frac{1}{d}}K^\times_{\infty,\le 1}.$$

The cardinality of the latter can be obtained by the Lipschitz principle; cf. last lecture.

By Dirichlet's unit theorem we have

$$\mathcal{O}_K^\times \cong \mu_K \oplus U, \qquad U \longleftrightarrow \mathrm{Log}_\infty(U)=\mathrm{Log}_\infty(\mathcal{O}_K)$$

where U is free abelian generated by $r-1$ elements. As $\mathcal{O}_K^\times \circlearrowright K$ acts freely, we have $|\underline{\mathfrak{a}}_X/\mathcal{O}_K| = \frac{1}{|\mu_K|}\,|\underline{\mathfrak{a}}_X/U|$.

Lemma:

Let $F_{\leq X\mathrm{Nr}(\underline{a})}$ be a fundamental domain for $U \curvearrowright K^\times_{\infty, \leq X\mathrm{Nr}(\underline{a})}$. Then

$$|\underline{a}_X/U| = |\sigma_\infty(\underline{a}) \cap F_{\leq X\mathrm{Nr}(\underline{a})}|$$

Proof: Let $y \in \underline{a}_X/U$, i.e., $y = aU$ for some $a \in \underline{a}_X$. Then

$$a \in \underline{a}_X \Leftrightarrow \sigma_\infty(a) \in \bigsqcup_{\varepsilon \in U} \sigma_\infty(\varepsilon^{-1}) F_{\leq X\mathrm{Nr}(\underline{a})}$$

$$\Leftrightarrow \exists! \varepsilon \in U \quad \sigma_\infty(a\varepsilon) \in F_{\leq X\mathrm{Nr}(\underline{a})}.$$

Hence $\sigma_\infty(y) \cap F_{\leq X\mathrm{Nr}(\underline{a})}$ is a singleton and we obtain an injective map

$$\psi : \underline{a}_X/U \longrightarrow \sigma_\infty(\underline{a}) \cap F_{\leq X\mathrm{Nr}(\underline{a})}$$

Now suppose $v \in \sigma_\infty(\underline{a}) \cap F_{\leq X\mathrm{Nr}(\underline{a})}$. Let $a \in \underline{a}$ s.t. $v = \sigma_\infty(a)$. Since

$$F_{\leq X\mathrm{Nr}(\underline{a})} \subseteq K^\times_{\infty, \leq X\mathrm{Nr}(\underline{a})}$$

we obtain $a \neq 0$ and

$$|\mathrm{Nr}_{K/\mathbb{Q}}(a)| = \mathrm{Nr}(\sigma_\infty(a)) \leq X\mathrm{Nr}(\underline{a}),$$

i.e., $aU \in \underline{a}_X/U$ and, thus, $v \in \psi(\underline{a}_X/U)$, i.e., ψ is onto. $\blacksquare$

simplify notation

We compute a fundamental domain for $U \circlearrowright K^{\times}_{\infty,\leq X}$, $(\varepsilon, v) \longmapsto \sigma_\infty(\varepsilon) v$ (and then divide the output of Lipschitz by vol. of the f.d.)

Suppose $\mathcal{B} = (\varepsilon_1, \dots, \varepsilon_{r-1})$ is a set of fundamental units, i.e.

- $U = \langle \varepsilon_1, \dots, \varepsilon_{r-1} \rangle$, and
- $\mathrm{Log}_\infty \mathcal{B} = (\mathrm{Log}_\infty(\varepsilon_1), \dots, \mathrm{Log}_\infty(\varepsilon_{r-1}))$ is a basis of $\mathrm{Log}_\infty \mathcal{O}_K^\times < H(\mathbb{R})$.

Let

$$\mathcal{P} = \mathcal{P}_{\mathrm{Log}_\infty \mathcal{B}} = \left\{ \sum_{i=1}^{r-1} t_i \mathrm{Log}_\infty(\varepsilon_i) : t_i \in [0,1[\right\} \leftarrow \text{f.d. for } U \circlearrowright H(\mathbb{R}).$$

Recall: $\mathrm{Log} : K_\infty^\times \longrightarrow \mathbb{R}^{r_1+r_2}$, $(x_1, \dots, x_{r_1}, z_1, \dots, z_{r_2}) \longmapsto (\log|x_1|, \dots, \log|x_{r_1}|, 2\log|z_1|, \dots, 2\log|z_{r_2}|)$

Lemma:

$F_1 := \mathrm{Log}^{-1}(\mathcal{P})$ is a f.d. for $\mathcal{U} \curvearrowright K_{\infty,1}^\times$

Pf:

$\sum \mathrm{Log}(v)_i = 0$

- $v \in K_{\infty,1}^\times \Rightarrow \mathrm{Log}(v) \in H(\mathbb{R}) \Rightarrow \mathrm{Log}(v) = \mathrm{Log}_\infty(\varepsilon) + \omega$ for some $\omega \in \mathcal{P}$ and some $\varepsilon \in \mathcal{U}$

$$\Rightarrow v = \sigma_\infty(\varepsilon)\mathrm{Log}^{-1}(\omega) \in \sigma_\infty(\varepsilon) F_1$$

$$\Rightarrow K_{\infty,1}^\times = \bigcup_{\varepsilon \in \mathcal{U}} \sigma_\infty(\varepsilon) F_1 .$$

- Let $\varepsilon, \hat{\varepsilon} \in \mathcal{U}$, then $\sigma_\infty(\varepsilon) F_1 \cap \sigma_\infty(\hat{\varepsilon}) F_1 \neq \emptyset \Leftrightarrow \sigma_\infty(\varepsilon^{-1}\hat{\varepsilon}) F_1 \cap F_1 \neq \emptyset$.

$\Rightarrow \exists f \in F_1$ s.t. $\mathrm{Log}_\infty(\varepsilon^{-1}\hat{\varepsilon}) + \mathrm{Log} f \in \mathcal{P}$

$\Rightarrow \mathrm{Log}_\infty(\varepsilon^{-1}\hat{\varepsilon}) + \mathcal{P} \cap \mathcal{P} \neq \emptyset \Rightarrow \mathrm{Log}_\infty(\varepsilon^{-1}\hat{\varepsilon}) = 0 \Rightarrow \varepsilon = \hat{\varepsilon}$

$$\Rightarrow K_{\infty,1}^\times = \bigsqcup_{\varepsilon \in \mathcal{U}} \sigma_\infty(\varepsilon) F_1$$

Corollary:

The cone $F_{\leq X} :=]0, X^{1/d}].F_1 = \{tf : f \in F_1, 0 < t \leq X^{1/d}\}$ is an f.d. for $U \curvearrowright K^\times_{\infty, \leq X}$

Moreover, $F_{\leq X} \subseteq K_\infty$ is precompact with Lipschitz-boundary

Pf: . f.d.-property:

$K^\times_{\infty,\leq X} = \bigsqcup_{0<t\leq X} K^\times_{\infty,t}$ and as U preserves $K^\times_{\infty,t}$, we have

$$K^\times_{\infty,\leq X}/_{G_\infty(U)} = \bigsqcup_{0<t\leq X} K^\times_{\infty,t}/_{G_\infty(U)}.$$

Using homogeneity, $K^\times_{\infty,t} = t^{\frac{1}{d}} K_{\infty,1}$ and thus $t^{\frac{1}{d}}.F_1$ is an f.d. for $U \curvearrowright K^\times_{\infty,t}$.

· Precompactness:

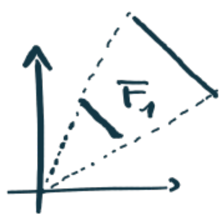
By Heine-Borel, it suffices to show that $F_{\leq X}$ is bdd. This follows if we can show that F_1 is bounded.

$\overline{F_1} = \overline{\mathrm{Log}^{-1}(\mathcal{P})} \subseteq \overline{\mathrm{Log}^{-1}(\overline{\mathcal{P}})} = \mathrm{Log}^{-1}(\overline{\mathcal{P}})$ is cpt. by continuity of $\mathrm{Log}^{-1} : \mathbb{R}^r \longrightarrow K_\infty$ and compactness of $\overline{\mathcal{P}}$, thus F_1 is bounded.

· Lipschitz boundary:

$\partial F_{\leq X} = X^{\frac{1}{d}} F_1 \cup \{0\} \cup \{[0, X^{1/d}]\mathrm{Log}^{-1}(\Phi) : \Phi \text{ a face of } \mathcal{P}\}$.

All of these are images under Lipschitz maps.



$t^d = X \Rightarrow t^{d-1} = X \cdot X^{-\frac{1}{d}} = X^{1-\frac{1}{d}}$

Corollary

Let $\underline{a} \in \mathcal{J}$, then

$$\left|\sigma_\infty(\underline{a}) \cap F_{\leq X \mathrm{Nr}(\underline{a})}\right| = \frac{2^{r_2}\mathrm{vol}(F_{\leq 1})}{|\mathrm{disc}(\mathcal{O}_K)|^{1/2}} X + O_{\underline{a}, F_{\leq 1}}\left(X^{1-\frac{1}{d}}\right)$$

Pf: Apply Lipschitz principle with $\Lambda = \sigma_\infty(\underline{a})$ and $\Omega := F_{\leq 1}$ (note: $\Omega_t = t\Omega = F_{\leq t^d}$) $\blacksquare$

Therefore:

$$\sum_{m \leq X} \left|\{\underline{a} \in \mathcal{J} : \mathrm{Nr}(\underline{a}) = m\}\right| = \sum_{[\underline{a}] \in \mathcal{Cl}(K)} \left|\{a \in \underline{a} - \{0\} : |\mathrm{Nr}(a)| \leq X \mathrm{Nr}(\underline{a})\} / \mathcal{O}_K^\times\right|$$

$$= \sum_{[\underline{a}] \in \mathcal{Cl}(K)} \left(\frac{2^{r_2}\mathrm{vol}(F_{\leq 1})}{|\mu_K|\, |\mathrm{disc}(\mathcal{O}_K)|^{1/2}} X + O_{\underline{a}, F_{\leq 1}}\left(X^{1-\frac{1}{d}}\right) \right)$$

$$= \frac{2^{r_2}\mathrm{vol}(F_{\leq 1})\, h(K)}{|\mu_K|\, |\mathrm{disc}(\mathcal{O}_K)|^{1/2}} X + O_K\left(X^{1-\frac{1}{d}}\right)$$

<u>Lemma</u>:

$$\mathrm{vol}(F_{\le 1}) = 2^{r_1}\pi^{r_2}\,\mathrm{reg}(\mathcal{O}_K)$$

Pf: Let $z\in K_\infty^\times$. We use polar coordinates:

$$z = (\pm x_1,\dots,\pm x_{r_1}, \rho_1 e(\vartheta_1),\dots,\rho_{r_2}e(\vartheta_{r_2})) \quad \text{w/ } x_i,\rho_i\in\mathbb{R}_{>0} \text{ and } \vartheta_i\in[0,1[\,.$$

Let $|\cdot|: K_\infty^\times \to \mathbb{R}^r_{>0}$, $|z| = (x_1,\dots,x_{r_1},\rho_1,\dots,\rho_{r_2})$ and let $\mathrm{Log}_{abs}:\mathbb{R}^r_{>0}\xrightarrow{\sim}\mathbb{R}^r$ be given by

$$\mathrm{Log}_{abs}(v_i)_{i=1}^{r_1+r_2} := (d_i\log v_i)_{i=1}^{r_1+r_2}.$$

Then the fibers of $|\cdot|$ are $\cong \{\pm1\}^{r_1}\times(S^1)^{r_2}$ and $\mathrm{Log}=\mathrm{Log}_{abs}\circ|\cdot|$, i.e. $\mathrm{Log}(z)=\mathrm{Log}_{abs}(|z|)$.

We let $\mathrm{Nr}_{abs}:\mathbb{R}^r_{>0}\longrightarrow\mathbb{R}$, $(x_1,\dots,x_{r_1},\rho_1,\dots,\rho_{r_2})\longmapsto \prod_{i=1}^{r_1}x_i\prod_{j=1}^{r_2}\rho_j^2$

<u>Claim</u>: $z\in F_{\le1} \Leftrightarrow \mathrm{Nr}(z)\le 1$ and $\mathrm{Log}(\mathrm{Nr}(z)^{-\frac{1}{d}}z)\in P$

Pf: Exercise.

"$\Rightarrow$" $z\in F_{\le1}\Leftrightarrow z\in\,]0,1]F_1 \Leftrightarrow \exists t\in\,]0,1]$ s.t. $t^{-1}z\in \overset{\mathrm{Log}^{-1}(P)}{\overset{\shortparallel}{F_1}} \Leftrightarrow \exists t\in\,]0,1]\ \exists p\in P$ s.t. $\mathrm{Log}(t^{-1}z)=p$

$\Rightarrow \mathrm{Nr}(z)=t^d\,\mathrm{Nr}(t^{-1}z) = t^d\exp\left(\sum_i \mathrm{Log}(t^{-1}z)_i\right) = t^d\exp\left(\underbrace{\sum_i p_i}_{=0}\right) = t^d\le 1$

and $\mathrm{Log}(\mathrm{Nr}(z)^{-\frac{1}{d}}z) = \mathrm{Log}((t^d)^{-\frac{1}{d}}z)=\mathrm{Log}(t^{-1}z)=p\in P$. $\leftarrow P\subseteq H(\mathbb{R})$

"$\Leftarrow$" $p=\mathrm{Log}(\mathrm{Nr}(z)^{-\frac{1}{d}}z)\in P \Rightarrow \mathrm{Nr}(z)^{-\frac{1}{d}}z\in F_1 \Rightarrow z\in \mathrm{Nr}(z)^{\frac{1}{d}}F_1 \underset{\mathrm{Nr}(z)\le 1}{\subseteq}\,]0,1]F_1$.

Claim: $|F_{\le 1}| =]0,1]\,\mathrm{Log}_{abs}^{-1}(\mathcal{P})$

Pf: Exercise.

$(x,y) \in |F_{\le 1}| \iff \exists z \in F_{\le 1}$ s.t. $(x,y) = |z|$

$$\iff \exists p\in\mathcal{P}\ \exists t\in\,]0,1] \text{ s.t. } (x,y) = |t\,\mathrm{Log}^{-1}(p)| = t\,|\mathrm{Log}^{-1}(p)|$$

Hence it suffices to show that $|\mathrm{Log}^{-1}(\mathcal{P})| = \mathrm{Log}_{abs}^{-1}(\mathcal{P})$.

"$\subseteq$" $(x,y)\in|\mathrm{Log}^{-1}(\mathcal{P})| \iff \exists z\in F_1$ s.t. $(x,y)=|z| \Rightarrow \mathrm{Log}_{abs}((x,y)) = \mathrm{Log}_{abs}(|z|) = \mathrm{Log}(z)\in\mathcal{P}$

$$\Rightarrow (x,y)\in \mathrm{Log}_{abs}^{-1}(\mathcal{P})$$

"$\supseteq$" Suppose $(x,y)\in\mathrm{Log}_{abs}^{-1}(\mathcal{P})$. $|\cdot|$ is onto, thus let $z\in K_\infty^\times$ s.t. $(x,y)=|z|$. Then

$$\mathrm{Log}(z)=\mathrm{Log}_{abs}(|z|)\in\mathcal{P}$$

$\Rightarrow (x,y)\in|\mathrm{Log}^{-1}(\mathcal{P})|$.

Claim: $z\in F_{\le 1} \iff \mathrm{Nr}_{abs}(|z|)\le 1$ and $\mathrm{Log}_{abs}\left(\mathrm{Nr}(|z|)^{-\frac{1}{d}}|z|\right)\in\mathcal{P} \iff |z|\in|F_{\le 1}|$

Pf: Exercise.

$\mathrm{Nr}(z)=\mathrm{Nr}(|z|)$ and thus $\mathrm{Log}\left(\mathrm{Nr}(z)^{-\frac{1}{d}} z\right) = \mathrm{Log}_{abs}\left(|\mathrm{Nr}(z)^{-\frac{1}{d}} z|\right) = \mathrm{Log}_{abs}\left(\mathrm{Nr}(z)^{-\frac{1}{d}}|z|\right)$

$$= \mathrm{Log}_{abs}\left(\mathrm{Nr}(|z|)^{-\frac{1}{d}}|z|\right).$$

Hence the first equivalence.

For the 2nd equivalence, we only need to check $|z|\in|F_{\le1}| \Rightarrow z\in F_{\le 1}$ ($z\in F_{\le1}\Rightarrow |z|\in|F_{\le1}|$ is obvious).

Supp. $|z|\in|F_{\le1}| \Rightarrow |z|\in\,]0,1]\,\mathrm{Log}_{abs}^{-1}(\mathcal{P}) \Rightarrow \exists t\in\,]0,1]$ s.t. $t^{-1}|z|\in \mathrm{Log}_{abs}^{-1}(\mathcal{P})$

$$\Rightarrow \exists t\in\,]0,1] \text{ s.t. } \mathrm{Log}_{abs}(t^{-1}|z|) = \mathrm{Log}_{abs}(|t^{-1}z|) = \mathrm{Log}(t^{-1}z)\in\mathcal{P}$$

$$\Rightarrow z\in\,]0,1]\,\mathrm{Log}^{-1}(\mathcal{P}) = F_{\le 1}.$$

Therefore

$$\mathrm{vol}(F_{\le 1}) = \sum_{(\varepsilon_1,\dots,\varepsilon_{r_1})\in\{\pm 1\}^{r_1}} \int_{\mathbb{R}_{>0}^{r_1,r_2}} (2\pi)^{r_2} \int_{[0,1[^{r_2}} \underbrace{\mathbb{1}_{F_{\le 1}}\left(((-1)^{\varepsilon_1}x_1,\dots,(-1)^{\varepsilon_{r_1}}x_{r_1},\varrho_1 e(\vartheta_1),\dots,e(\vartheta_{r_2})\right)}_{=\mathbb{1}_{|F_{\le 1}|}(x_1,\dots,x_{r_1},\varrho_1,\dots,\varrho_{r_2})} \varrho_1\cdots\varrho_{r_2}\, d\vartheta\, d(\underline{x},\underline{\varrho})$$

$z\in F_{\le 1} \Leftrightarrow |z|\in\,]0,1]\,\mathrm{Log}_{abs}^{-1}(P)$

$$= 2^{r_1}(2\pi)^{r_2} \int_{\mathbb{R}_{>0}^{r}} \mathbb{1}_{]0,1]\mathrm{Log}_{abs}^{-1}(P)}(\underline{x},\underline{\varrho})\, \varrho_1\cdots\varrho_{r_2}\, d(\underline{x},\underline{\varrho})$$

$$\mathrm{Log}_{abs}^{-1}(v)_i = \begin{cases} e^{v_i} & i=1,\dots,r_1 \\ e^{\frac{1}{2}v_i} & i=r_1+1,\dots,r_2 \end{cases}$$

Substitution

$$= 2^{r_1}(2\pi)^{r_2} \int_{\mathbb{R}^r} \mathbb{1}_{]0,1]\mathrm{Log}_{abs}^{-1}(P)}\left(\mathrm{Log}_{abs}^{-1}(v)\right) \exp\left(\frac{1}{2}\sum_{i=r_1+1}^{r_2} v_i\right) J_v\left(\mathrm{Log}_{abs}^{-1}\right) dv$$

$$J_v\left(\mathrm{Log}_{abs}^{-1}\right) = 2^{-r_2}\exp\left(\sum_{i=1}^{r_1} v_i + \frac{1}{2}\sum_{i=r_1+1}^{r_2} v_i\right)$$

$$= 2^{r_1}\pi^{r_2} \int_{\mathbb{R}^r} \mathbb{1}_{\mathrm{Log}_{abs}(]0,1]\mathrm{Log}_{abs}^{-1}(P))}(v)\exp(Tv)\, dv$$

$Tv = \sum_{i=1}^{r} v_i$

<u>Claim</u>: $v \in \mathrm{Log}_{abs}(]0,1]\,\mathrm{Log}_{abs}^{-1}(\mathcal{P})) \iff Tv \leq 0$ and $v - \frac{T(v)}{d}(1,\dots,1) \in \mathcal{P}$

<u>Pf</u>: "$\Rightarrow$" Let $p \in \mathcal{P}$ and $t \in]0,1]$. Then

$$\mathrm{Log}_{abs}(t\,\mathrm{Log}_{abs}^{-1}(p)) = (\log t)(1,\dots,1) + p.$$

Thus

$$Tv = \sum(\log t + p_i) \overset{p \in H(\mathbb{R})}{=} d(\log t) \leq 0$$

and

$$\mathrm{Log}_{abs}(t\,\mathrm{Log}_{abs}^{-1}(p)) - \frac{Tv}{d}(1,\dots,1) = p \in \mathcal{P}$$

"$\Leftarrow$" Let $p = v - \frac{T(v)}{d}(1,\dots,1)$, then

$$v = \frac{T(v)}{d}(1,\dots,1) + p = \mathrm{Log}_{abs}(\overbrace{e^{T(v)/d}}^{\in(0,1]}\,\mathrm{Log}_{abs}^{-1}(p))$$

$$\in \mathrm{Log}_{abs}(]0,1]\,\mathrm{Log}_{abs}^{-1}(\mathcal{P}))$$

Note:

- $\ker T = H(\mathbb{R})$
- $\underbrace{\frac{1}{\sqrt{d}}(1,\dots,1)}_{=:\omega} \perp H(\mathbb{R})$
- Let $P_H : \mathbb{R}^r \longrightarrow H(\mathbb{R})$ be the orthogonal projection, then

$$P_H(v) = v - \langle v, \omega\rangle \omega \qquad (\|\omega\| = 1$$

$$= v - \frac{T(v)}{\sqrt{d}}\,\omega .$$

- $T(v) = T\big(v - \langle v,\omega\rangle\omega\big) + T\big(\langle v,\omega\rangle\omega\big) = \langle v,\omega\rangle T(\omega) = \sqrt{d}\,\langle v,\omega\rangle$

- Thus for any $f : \mathbb{R}^r \longrightarrow \mathbb{C}$ we have

$$\int_{\mathbb{R}^r} f(v)\,dv = \int_{\mathbb{R}} \int_{H(\mathbb{R})} f(t\omega + h)\,dh\,dt$$

$$\operatorname{vol}(F_{\leq s}) = 2^{r_1}\pi^{r_2} \int_{\mathbb{R}^r} \mathbb{1}_{]-\infty,0]}(Tv)\, \mathbb{1}_{\mathcal{P}}\left(v - \frac{T(v)}{\sqrt{d}}\omega\right) e^{T(v)}\, dv$$

$$T(t\omega) = t\sqrt{d}$$

$$= 2^{r_1}\pi^{r_2} \int_{-\infty}^{0} e^{t} \left(\int_{H(\mathbb{R})} \mathbb{1}_{\mathcal{P}}\Big(\underbrace{t\omega + h - \frac{T(t\omega+h)}{\sqrt{d}}\omega}_{=h}\Big)\, dh \right) dt$$

$$= 2^{r_1}\pi^{r_2} \operatorname{vol}_{H(\mathbb{R})}(\mathcal{P}) \int_{-\infty}^{0} e^{t}\, dt = 2^{r_1}\pi^{r_2} \operatorname{reg}(K)$$

∎