The class number formula

Theorem

Let $K/\mathbb{Q}$ a #-field, $[K:\mathbb{Q}] = d = r_1 + 2r_2$. Then

$$\left|\{\mathfrak{a} \trianglelefteq \mathcal{O}_K : Nr(\mathfrak{a}) \leq X\}\right| = \frac{2^{r_1}(2\pi)^{r_2} h(K)\,\mathrm{reg}(K)}{\omega_K |\mathrm{disc}(K)|^{1/2}} X + O_K\left(X^{1-\frac{1}{d}}\right),$$

where $h(K) = |Cl(K)|$, $\omega_K = |\mu_K|$.

Corollary:

Let $K/\mathbb{Q}$ imaginary, quadratic of discriminant $D \ll -1$. Then

$$h(K) = \lim_{X\to\infty} \frac{\sqrt{-D}}{\pi} \frac{\left|\{\mathfrak{a} \trianglelefteq \mathcal{O}_K : Nr(\mathfrak{a}) \leq X\}\right|}{X}.$$

Excursion - The Dedekind ζ-function and the class number formula

Proposition:

Let $K/\mathbb{Q}$ a number field. The series

$$\zeta_K(s) = \sum_{\underline{\mathfrak{a}} \trianglelefteq \mathcal{O}_K} \frac{1}{\mathrm{Nr}(\underline{\mathfrak{a}})^s}$$

converges absolutely on $\{s \in \mathbb{C} : \mathrm{Re}(s) > 1\}$ and there it defines a holomorphic function

Remark: Let $r_K(m) = |\{\underline{\mathfrak{a}} \trianglelefteq \mathcal{O}_K : \mathrm{Nr}(\underline{\mathfrak{a}}) = m\}|$. Then

$$\zeta_K(s) = \sum_{m \in \mathbb{N}} \frac{r_K(m)}{m^s}.$$

Proof (sketch): Recall that $N_K(X) = \sum_{m \leq X} r_K(m) \ll_\varepsilon X^{1+\varepsilon}$, hence integration by parts / Abel summation yields

$$\sum_{m \leq X} \frac{r_K(m)}{m^s} = \left[x^{-\sigma} N_K(x)\right]_{\frac{1}{2}}^{X} - \int_{1/2}^{X} N_K(x) \frac{\sigma}{x^{\sigma+1}}\, dx \ll_\varepsilon X^{1+\varepsilon-\sigma} + \sigma \int_{1/2}^{X} x^{\varepsilon-\sigma}\, dx \ll_\varepsilon 1$$

for $\sigma > 1+\varepsilon$. $\blacksquare$

Definition

The function $\sigma \longmapsto \zeta_K(\sigma)$ is the Dedekind ζ-function.

Theorem

The Dedekind ζ-function admits meromorphic continuation to the half-plane $\{s\in\mathbb{C} : \mathrm{Re}(s) > 1-\frac{1}{d}\}$ with simple pole at $s=1$ and

$$\mathrm{res}_{s=1}\zeta_K = \underbrace{\frac{2^{r_1}(2\pi)^{r_2}h(\mathcal{O}_K)\mathrm{reg}(\mathcal{O}_K)}{\omega_K|\mathrm{disc}(K)|^{1/2}}}_{\varrho}$$

Proof: Let ϱ be the RHS and $r_{K,0}(m) := r_K(m) - \varrho$. By the class number formula

$$\sum_{m\leq X} r_{K,0}(m) = \mathcal{O}(X^{1-\frac{1}{d}}).$$

Using integration by parts / Abel summation again, one shows that for any $\eta > 0$ the series

$$\sum_{m\geq 1} \frac{r_{K,0}(m)}{m^s}$$

converges absolutely and uniformly on $\{s\in\mathbb{C} : \mathrm{Re}(s) > 1-\frac{1}{d}+\eta\}$ and there defines a hol. fct.

For all $X\geq 1$

truncated fcts.

$$\sum_{m\geq 1}\frac{r_{K,0}(m)}{m^s} = \zeta_{K,X}(s) - \varrho\,\zeta_X(s)$$

The last term converges uniformly on $\{s\in\mathbb{C} : \mathrm{Re}(s) > 1+\eta\}$ with meromorphic continuation to $\{s\in\mathbb{C} : \mathrm{Re}(s) > 0\}$ and unique simple pole at $s=1$ with residue 1. Hence

$$\zeta_K(s) = \varrho\,\zeta(s) + \sum_{m\geq 1}\frac{r_{K,0}(m)}{m^s}$$

Hence ζ_K admits meromorphic continuation to $\{s\in\mathbb{C} : \mathrm{Re}(s) > 1-\frac{1}{d}\}$ with unique simple pole at $s=1$. ∎

Towards the proof of the class number formula:

Lemma:

Let $\mathfrak{a} \in \mathcal{I}$. Define $\mathfrak{a} - \{0\} \longrightarrow \mathcal{I}$ by $a \longmapsto a.\mathfrak{a}^{-1}$. For any $X > 1$, this map gives a bijection

$$\left\{ a \in \mathfrak{a} - \{0\} : \mathrm{Nr}(a) \leq X \, \mathrm{Nr}(\mathfrak{a}) \right\} \big/ \mathcal{O}_K^\times \longleftrightarrow \left\{ \mathfrak{b} \in [\mathfrak{a}^{-1}] \cap \mathcal{J} : \mathrm{Nr}(\mathfrak{b}) \leq X \right\}.$$

Proof: cf. finiteness of $\mathcal{C}\ell(K)$.

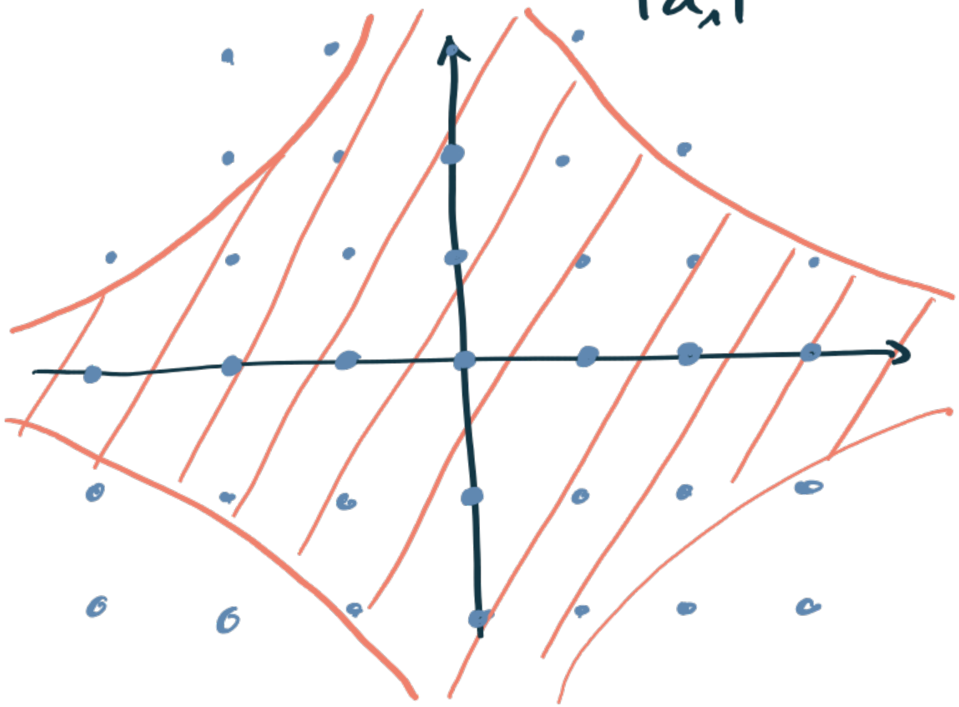
Corollary

$$\left|\left\{ \mathfrak{a} \triangleleft \mathcal{O}_K : \mathrm{Nr}(\mathfrak{a}) \leq X \right\}\right| = \sum_{g \in \mathcal{C}\ell(K)} \left| \left\{ \mathfrak{b} \in g^{-1} \cap \mathcal{J}_K : \mathrm{Nr}(\mathfrak{b}) \leq X \right\} \right|$$

$$= \sum_{[\mathfrak{a}] = g \in \mathcal{C}\ell(K)} \left| \underbrace{\left\{ a \in \mathfrak{a} - \{0\} : |\mathrm{Nr}(a)| \leq X \, \mathrm{Nr}(\mathfrak{a}) \right\}}_{\substack{\cong \, \# \text{ of lattice points} \\ \text{in a (non-convex) domain} \\ \rightsquigarrow \text{ counting problem.}}} \big/ \mathcal{O}_K^\times \right|$$

Example: Let $K=\mathbb{Q}(\sqrt{D})$, $Nr(a)=C$, $\sigma_\infty(a)=(a_1,a_2)$.

Then $|Nr(a)|\leq CX \Leftrightarrow |a_2|\leq \dfrac{CX}{|a_1|}$



Definition:

Let $Nr: K_\infty^\times \longrightarrow \mathbb{R}_{>0}$, $Nr(v)=\prod_{i=1}^{r_1+r_2}|v_i|^{d_i}$

Then
- $Nr(\sigma_\infty(z))=|Nr_{K|\mathbb{Q}}(z)|$
- Nr is $\mathcal{O}_K^\times$-invariant: $Nr(\sigma_\infty(\varepsilon)v)=Nr(v)$
- Homogeneous of degree $d=[K:\mathbb{Q}]$ (for $\lambda\in\mathbb{R}_{>0}$)
- $K_{\infty,X}^\times := \{v\in K_\infty^\times : Nr(v)=X\} = X^{\frac{1}{d}}K_{\infty,1}^\times$ and $K_{\infty,\leq X}^\times := \bigcup_{0<X'\leq X} K_{\infty,X'}^\times = X^{\frac{1}{d}}K_{\infty,\leq 1}^\times$.

<u>Goal</u>: Given $\sigma_\infty(\underline{a}) \subseteq K_\infty$, count the number of $\mathcal{O}_K^\times$-orbits contained in the domain given by $\{v \in K^\infty : Nr(v) \leq X\, Nr(\underline{a})\}$.

<u>Method</u>: Find nice fundamental domain $\mathcal{F}$ for $\mathcal{O}_K^\times \curvearrowright K^\times_{\infty, \leq X Nr(\underline{a})}$ and count pts. in $\mathcal{F}$.

The Lipschitz principle

Let $\Omega \subseteq \mathbb{R}^d$ compact, $\Lambda < \mathbb{R}^d$ a lattice, $t > 0$, $\Omega_t := t\Omega$. We want to evaluate

$$N_\Omega(t, \Lambda) = |\Lambda \cap \Omega_t| \quad \text{as } t \to \infty.$$

Conjecture: Let $\Lambda = \mathbb{Z}^2$, $\Omega = \{(x,y) : x^2 + y^2 < 1\}$. Then $N_\Omega(t, \Lambda) = \pi r^2 + O_\varepsilon(r^{\frac{1}{2}+\varepsilon})$.

Best known: $\frac{1}{2} + \varepsilon \leq \frac{131}{208}$.

Theorem (Lipschitz principle)

Suppose $\partial\Omega$ is Lipschitz, i.e., $\exists$ finitely many $\varphi_i : [0,1]^{d-1} \longrightarrow \mathbb{R}^d$ Lipschitz s.t.

$$\partial\Omega = \bigcup_{i=1}^{m} \varphi_i([0,1]^{d-1}).$$

Then $N_\Omega(t, \Lambda) = \frac{\mathrm{vol}(\Omega)}{\mathrm{covol}(\Lambda)} t^d + O_{\Lambda,\Omega}(t^{d-1})$.

Proof: Case 1: $\Lambda = \mathbb{Z}^d$.

Given $\lambda \in \Lambda$, let $C(\lambda) = \prod_{i=1}^{d} [\lambda_i, \lambda_i + 1[$, so that $C(\lambda)$ is a fundamental domain for $\Lambda \curvearrowright \mathbb{R}^d$ containing λ. Thus $\Omega_t = \bigsqcup_\lambda (\Omega_t \cap C(\lambda))$ and

$$\underbrace{|\{\lambda \in \Lambda : C(\lambda) \subseteq \Omega_t\}|}_{=\alpha(t)} \leq \underbrace{|\Lambda \cap \Omega_t|}_{=N_\Omega(t,\Lambda)} \leq \underbrace{|\{\lambda \in \Lambda : C(\lambda) \cap \Omega_t = \emptyset\}|}_{=\beta(t)}$$

Note: $\alpha(t) \leq \mathrm{vol}(\Omega_t) \leq \beta(t)$

We prove that $\exists C>0$ s.t. $\beta(t)-\alpha(t)\leq C t^{d-1}$, i.e.,

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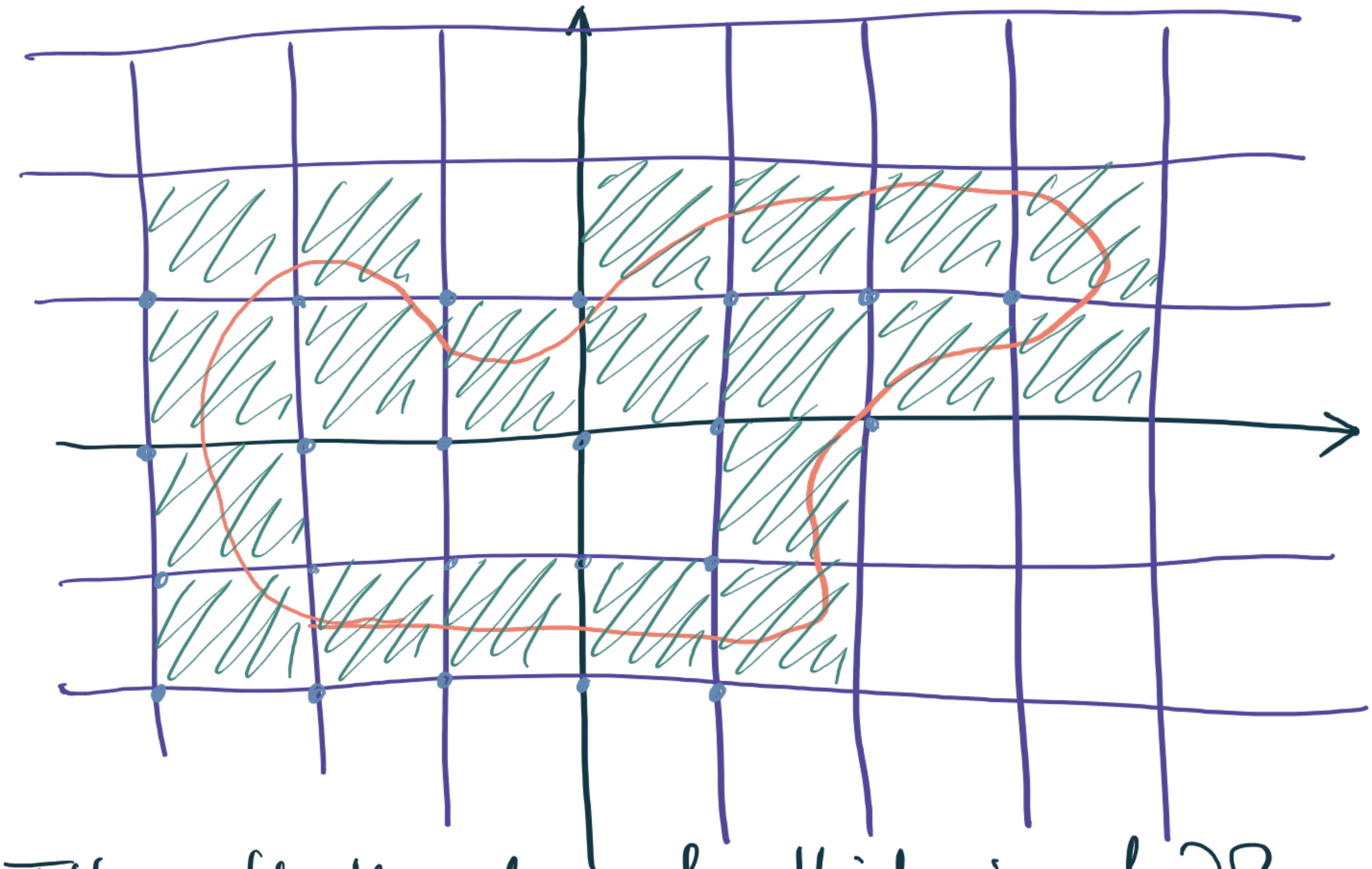
$$N_\Omega(t,\Lambda)=\mathrm{vol}(\Omega_t)+O(t^{d-1})=\mathrm{vol}(\Omega)t^d+O(t^{d-1})$$

What is $\beta(t)-\alpha(t)$?



It's roughly the volume of a thickening of $\partial\Omega_t$.

Note: $\lambda \in \{\lambda \in \mathbb{Z}^d : C(\lambda) \cap \Omega_t \neq \emptyset\} - \{\lambda \in \mathbb{Z}^d : C(\lambda) \subseteq \Omega_t\}$

$$\Rightarrow d(\lambda, \partial\Omega_t) \leq \operatorname{diam}(C(\lambda)) = \sqrt{d} = (1 + \cdots + 1)^{\frac{1}{2}}$$

Thus we only have to bound the # of lattice pts. within $\sqrt{d}$ of $\partial\Omega_t$.

We will find $O(t^{d-1})$-many $\{v_1, \ldots, v_s\} \subseteq \partial\Omega_t$ s.t. every pt. in $\partial\Omega_t$ is within $M = O_\Omega(1)$ of some v_i. As

$$|B_R(v) \cap \mathbb{Z}^d| \ll_R 1 \quad (v \in \mathbb{R}^d, R > 0),$$

the claim then follows.

$\ulcorner |\{\lambda \in \mathbb{Z}^d : C(\lambda) \cap \Omega_t \neq \emptyset\} - \{\lambda \in \mathbb{Z}^d : C(\lambda) \subseteq \Omega_t\}|$

$$\leq \left| \bigcup_{i=1}^{s} \left(B_{\sqrt{d}+M}(v_i) \cap \mathbb{Z}^d\right) \right| \ll_{d,\Omega} s \ll t^{d-1}$$

$\lrcorner$

To this end, let $\varphi_1, \dots, \varphi_m : [0,1]^{d-1} \longrightarrow \partial\Omega$ be the Lipschitz maps parametrizing the boundary. Then $\{t\varphi_i : i = 1, \dots, m\}$ parametrize $\partial\Omega_t = t\partial\Omega$.

Let

$$\mathcal{R} = \left\{ t\varphi_i\left(\frac{n}{t}\right) : n \in \mathbb{Z}^{d-1} \cap [0,t[^{d-1} \right\}.$$

Then $|\mathcal{R}| \leq m t^{d-1}$ and every pt. on $\partial\Omega_t$ is within $M = \max_i \operatorname{Lip}(\varphi_i)$ of a point in $\mathcal{R}$ (*). This completes this case.

(*) $v \in \partial\Omega_t \Rightarrow v = t\varphi_i(x)$ for some i and for some $x \in [0,1[^{d-1}$. Let $n \in \mathbb{Z}^{d-1}$ s.t. $\|x - \frac{n}{t}\| \leq \frac{1}{t}$, then

$$\left\|v - t\varphi_i\left(\frac{n}{t}\right)\right\| = t\left\|\varphi_i(x) - \varphi_i\left(\frac{n}{t}\right)\right\| \leq t \operatorname{Lip}(\varphi_i) \left\|x - \frac{n}{t}\right\| \leq M.$$

<u>Case 2:</u> General $\Lambda = \mathbb{Z}^d g$ $(g \in GL_d(\mathbb{R}))$. Then

$N_\Omega(t,\Lambda) = N_{\Omega g^{-1}}(t, \mathbb{Z}^d)$. If $\{\varphi_1, \dots, \varphi_m\}$ parametrize $\partial\Omega$, then $\tilde{\varphi}_i = g^{-1} \circ \varphi_i$ are Lipschitz and parametrize $\partial(\Omega g^{-1}) = \partial\Omega g^{-1}$. Moreover,

- $\operatorname{Lip}(\tilde{\varphi}_i) \leq \|g^{-1}\| \operatorname{Lip}(\varphi_i)$
- $\dfrac{\operatorname{vol}(\Omega)}{\operatorname{covol}(\Lambda)} = \operatorname{vol}(\Omega g^{-1})$.

$$\therefore \left| N_\Omega(t,\Lambda) - \frac{\mathrm{vol}(\Omega)}{\mathrm{covol}(\Lambda)} t^d \right| = \left| N_{\Omega g^{-1}}(t, \mathbb{Z}^d) - \mathrm{vol}(\Omega g^{-1}) t^d \right|$$

$$\ll_d \left(\max \mathrm{Lip}(\tilde{\varphi}_i) \right) t^{d-1}$$

take infimum of $\|g^{-1}\|$ over g s.t. $\Lambda = \mathbb{Z}^d g$.

$$\ll_{\Lambda,\Omega} t^{d-1}.$$

$\blacksquare$