

1. Let Q be a number field, K/Q a finite extension of degree $d = [K : Q]$, $A = \mathcal{O}_Q$, and $B = \mathcal{O}_K$.
 - a) Show that $B = \mathcal{O}_K(A)$.
 - b) Let $\mathbf{z}, \mathbf{z}' \in B^d$ and suppose that every element of $\mathbf{z}'$ can be expressed as a linear combination of the elements of $\mathbf{z}$. Show that there exists $a \in A$ such that

$$\text{disc}_{B|A}(\mathbf{z}') = a^2 \text{disc}_{B|A}(\mathbf{z}).$$

- c) Let $\mathfrak{p} \triangleleft A$ be a non-zero prime ideal, let $A_{\mathfrak{p}} \subseteq Q$ be the localization of A at $\mathfrak{p}$, and $B_{\mathfrak{p}} = \mathcal{O}_K(A_{\mathfrak{p}})$. Let $\mathfrak{D}_{B|A}$ and $\mathfrak{D}_{B_{\mathfrak{p}}|A_{\mathfrak{p}}}$ denote the discriminant ideals corresponding to the extensions $B|A$ and $B_{\mathfrak{p}}|A_{\mathfrak{p}}$ respectively. Show that

$$\mathfrak{D}_{B_{\mathfrak{p}}|A_{\mathfrak{p}}} = \mathfrak{D}_{B|A} \cdot A_{\mathfrak{p}}.$$

2. Suppose that K is a finite dimensional commutative algebra over the field k and let $d = \dim_k(K)$. Denote by $[\times \bullet]_{K/k} : K \rightarrow \text{End}_k(K)$ the k -linear map given by

$$\forall z, \xi \in K \quad [\times z]_{K/k}(\xi) = z\xi.$$

Suppose that K is non-reduced. Show that

$$\forall \xi = (\xi_1, \dots, \xi_d) \in K^d \quad \det(\text{tr}[\times \xi_i \xi_j]_{K/k}) = 0.$$

Hint: Let $\langle \bullet, \bullet \rangle_{K/k}$ be the trace pairing defined by

$$\forall z, \xi \in K \quad \langle z, \xi \rangle_{K/k} = \text{tr}[\times z \xi]_{K/k}.$$

Show that the homomorphism $K \rightarrow K^*$ given by $z \mapsto \langle z, \bullet \rangle_{K/k}$ has non-trivial kernel.

3. Let K/Q be a Galois extension. Let $\mathfrak{p} \in \text{Spec}(A)$ be a non-zero prime ideal.
 - a) Prove that the inertia degrees and ramification indices of the primes above $\mathfrak{p}$ are constant, i.e., there exist $e, f \in \mathbb{N}$ such that $e_{\mathfrak{P}/\mathfrak{p}} = e$ and $f_{\mathfrak{P}/\mathfrak{p}} = f$ for all $\mathfrak{P} \in \text{Spec}_{\mathfrak{p}}(B)$.
 - b) Prove that if the decomposition group $D_{\mathfrak{P}/\mathfrak{p}}$ for a $\mathfrak{P} \in \text{Spec}_{\mathfrak{p}}(B)$ is trivial, $\mathfrak{p}$ is totally split, i.e., $e_{\mathfrak{P}/\mathfrak{p}} = 1$ and $f_{\mathfrak{P}/\mathfrak{p}} = 1$ for all $\mathfrak{P} \in \text{Spec}_{\mathfrak{p}}(B)$.
 - c) Prove that if the decomposition group $D_{\mathfrak{P}/\mathfrak{p}}$ for a $\mathfrak{P} \in \text{Spec}_{\mathfrak{p}}(B)$ is the full Galois group, $\mathfrak{p}$ is totally ramified, i.e., $|\text{Spec}_{\mathfrak{p}}(B)| = 1$.