1. Let K be a number field and $\mathcal{O}_K \subseteq K$ the maximal order. Show that $\mathcal{O}_K = \mathcal{O}_K(\mathbb{Z})$.
2. Suppose that A is Dedekind. Show that ideals $\mathfrak{a}, \mathfrak{b} \lhd A$ are coprime, i.e., $A = \mathfrak{a} + \mathfrak{b}$, if and only if $\mathfrak{a}$ and $\mathfrak{b}$ don't have any common prime divisors.
3. Let A a Dedekind domain with field of fractions Q, K/Q a finite separable extension, and let $\mathfrak{p} \lhd A$ a non-zero prime ideal. Let $\mathfrak{P} \lhd \mathcal{O}_K(A)$ a prime ideal dividing $\mathfrak{p}.\mathcal{O}_K(A)$. Show that
$$\forall 0 < e \leqslant v_{\mathfrak{P}}\big(\mathfrak{p}.\mathcal{O}_K(A)\big) \qquad \mathfrak{p} = \mathfrak{P}^e \cap A.$$
4. Let $K/\mathbb{Q}$ be a number field of degree d, let θ be an algebraic integer of degree d, and let
$$P(X) = X^d + a_{d-1}X^{d-1} + \ldots + a_1 X + a_0$$
be its minimal polynomial. Furthermore, suppose that P is Eisenstein with respect to the prime p, that is
$$p \mid a_j \quad \text{for} \quad 0 \leqslant j \leqslant d-1 \qquad \text{and} \qquad p^2 \nmid a_0.$$
The goal of this exercise is to show that then $p \nmid |\mathcal{O}_K/\mathbb{Z}[\theta]|$.
   a) Assume to the contrary that p divides $|\mathcal{O}_K/\mathbb{Z}[\theta]|$. Show that in this case we can find $\xi \in \mathcal{O}_K$, such that $p\xi \in \mathbb{Z}[\theta]$ and $\xi \notin \mathbb{Z}[\theta]$.
   Hint: Every finite abelian group of order divisible by p contains an element of order p.
   b) Write
   $$p\xi = b_0 + b_1\theta + \ldots + b_{d-1}\theta^{d-1} \quad \text{with} \quad b_i \in \mathbb{Z},$$
   and let j be the smallest index such that $p \nmid b_j$. Prove that $b_j\theta^{d-1} \in p\mathcal{O}_K$.
   Hint: Since P is Eisenstein at p, we know that $\theta^d \in p\mathcal{O}_K$.
   c) Show that $\mathrm{N}_{K/\mathbb{Q}}\left(b_j\theta^{d-1}/p\right) \notin \mathbb{Z}$.
   d) Conclude by finding a contradiction.
5. Let p be a prime, let $\ell \geqslant 1$, let ζ be a primitive p^ℓ-th root of unity, and let K be the cyclotomic field $K := \mathbb{Q}(\zeta)$. In this exercise we want to determine the ring of integers of K.
   a) Show that
   $$\Phi(X) := \frac{X^{p^\ell} - 1}{X^{p^{\ell-1}} - 1} \in \mathbb{Z}[X]$$
   is the minimal polynomial of ζ.
   Hint: Use the Eisenstein criterion at p for $\Phi(X+1)$. To this end, show that for the reduction $\overline{\Phi}(X+1) \in \mathbb{F}_p[X]$ of $\Phi(X+1)$ we have
   $$\overline{\Phi}(X+1)X^{p^{\ell-1}} = X^{p^\ell}.$$
   b) Let $\xi := \zeta^{p^{\ell-1}}$. Prove that
   $$\left|\mathrm{N}_{\mathbb{Q}(\xi)/\mathbb{Q}}(\xi - 1)\right| = p \quad \text{and} \quad \left|\mathrm{N}_{K/\mathbb{Q}}(\xi - 1)\right| = p^{p^{\ell-1}}.$$
   c) Verify that
   $$(\xi - 1)\Phi'(\zeta) = p^\ell \zeta^{-1}.$$

d) Prove that

$$\left|\operatorname{disc}_{K/\mathbb{Q}}\left(1,\zeta,\zeta^2,\ldots,\zeta^{\phi(p^\ell)-1}\right)\right| = p^s \quad \text{with} \quad s := p^{\ell-1}(\ell p - \ell - 1).$$

Hint: Look at Sheet 6.

e) Let $q \in \mathbb{N}$ prime coprime to p. Show that q doesn't divide $|\mathcal{O}_K/\mathbb{Z}[\zeta]|$.
Hint: Use Sheet 2.

f) Conclude that $\mathcal{O}_K = \mathbb{Z}[\zeta]$.
Hint: You may want to consider $\mathbb{Z}[\zeta] = \mathbb{Z}[\zeta - 1]$ and use the conclusion of the previous exercise.