1. Let A be a domain, K a field, and suppose that $A \subseteq K$. Show that the integral closure $\mathcal{O}_K(A)$ of A in K is integrally closed.
2. Let $K = \mathbb{Q}(\sqrt{-5})$. Show that $\mathcal{O}_K$ is not a P.I.D.
   Hint: Is the ideal $(2, 1+\sqrt{-5})$ principal?
3. Let K be a number field and L/K a separable field extension of degree n. By the primitive element theorem, we know that $L = K(\alpha)$ for some $\alpha \in L$ with minimal polynomial $f \in K[X]$ of degree n. Let $\overline{K}$ be an algebraic closure of K. Denote by $\sigma_1, \ldots, \sigma_n$ an enumeration of $\mathrm{Hom}_K(L, \overline{K})$, and set
   $$\alpha_1 = \sigma_1(\alpha), \ldots, \alpha_n = \sigma_n(\alpha).$$
   Note that $\alpha_1, \ldots, \alpha_n$ are exactly the roots of f (which we know to be pairwise distinct since L/K is seperable).

   Given $a \in L$, let $[\times a]_{L/K} \colon L \to L$ be the K-linear map given by $[\times a]_{L/K}(b) = ab$ for all $b \in L$. Define $B \colon L \times L \to K$ by
   $$\forall a, b \in L \quad B(a,b) = \mathrm{tr}\big([\times ab]_{L/K}\big).$$
   We call B the trace-pairing form on L/K.
   a) Show that B is K-bilinear.
   b) Show that
   $$\det\big(B(\alpha_i, \alpha_j)\big) = \prod_{i<j} (\alpha_i - \alpha_j)^2.$$
   Conclude that the trace pairing form on L/K is non-degenerate.
   c) Prove that
   $$\begin{aligned}\det\big(B(\alpha_i, \alpha_j)\big) &= (-1)^{\frac{n(n-1)}{2}} \mathrm{Nr}_{L/K}\big(f'(\alpha)\big) \\ &= (-1)^{\frac{n(n-1)}{2}} \prod_{i=1}^{n} \sigma_i\big(f'(\alpha)\big).\end{aligned}$$
4. Let K be a number field and $\mathcal{B}$ be a $\mathbb{Q}$-basis of K contained in $\mathcal{O}_K$. Prove that if $\Delta(\mathcal{B})$ is square free, then $\mathcal{B}$ is a $\mathbb{Z}$-basis of $\mathcal{O}_K$.