1. In this exercise we recall the notion of Noetherian rings.
   a) Let A be a ring and M an A-module. Prove that the following are equivalent.
      - Every submodule of M is of finite type.
      - Every increasing sequence of submodules of M eventually stabilizes.
      - Every non-empty collection of submodules of M has a maximal element.

      An A-module is called *noetherian* if it has any of these three properties. A ring A is noetherian if it is noetherian as an A-module.
   b) Suppose that A is a noetherian ring and M is an A-module of finite type. Show that M is noetherian.
2. In this exercise, we will recall localization of rings. Let A be a ring and $S \subseteq A$ be a multiplicative subset i.e. a subset closed under multiplication. We define a relation on $A \times S$, $\sim$ by
$$(a, s) \sim (b, t) \iff \exists u \in S \text{ s.t. } u(at - bs) = 0.$$
   a) Show that $\sim$ defines an equivalence relation on $A \times S$.

   We call $A \times S/\sim$ the localization of A by S and denote it by $S^{-1}A$. The equivalence class of (a, s) in $S^{-1}A$ is often denoted $\frac{a}{s}$.
   b) Equip $S^{-1}A$ with a ring structure s.t. the map
$$\psi_S : A \to S^{-1}A$$
      defined by
$$a \mapsto \frac{a}{1}$$
      is a ring homomorphism.
   c) Prove the universal property of localisation: Given a morphism of rings
$$\phi : A \to B$$
      s.t. $\phi(S) \subseteq B^\times$ there is a unique morphism of rings
$$\tilde{\phi} : S^{-1}A \to B$$
      s.t.
$$\phi = \tilde{\phi} \circ \psi_S$$
   d) Let I be an ideal in A and S' be the image of S in A/I. Prove that
$$(S')^{-1}A/I \simeq S^{-1}A/S^{-1}I.$$
      $S^{-1}I$ is defined subsequently.
   e) Show that the map $\mathfrak{q} \mapsto \mathfrak{q} \cap A$ defines a map between prime ideals in $S^{-1}A$ and prime ideals of A which have empty intersection with S. Show that the inverse map is given by $\mathfrak{p} \mapsto \mathfrak{p}S^{-1}A$.

   We can also localise modules: Let M be an A-module. We define a relation on $M \times S$, $\sim$ by
$$(m, s) \sim (n, t) \iff \exists u \in S \text{ s.t. } u(tm - sn) = 0.$$
   This is also an equivalence relation. We call $M \times S/\sim$ the localization of M by S and denote it by $S^{-1}M$. The equivalence class of (m, s) in $S^{-1}M$ is denoted $\frac{m}{s}$.
   f) Equip $S^{-1}M$ with a $S^{-1}A$ module structure which extends the A-module structure on M.

g) Prove that given A-modules $N \subseteq M$. $S^{-1}(M/N) \simeq (S^{-1}M)/(S^{-1}N)$.

h) Prove that any submodule $N' \subseteq S^{-1}M$ is of the form $S^{-1}N$ for some submodule $N \subseteq M$. Conclude that all ideals of $S^{-1}A$ are of the form $S^{-1}I$ for some ideal $I \lhd A$.

i) Let $\mathfrak{p} \lhd A$ be a prime ideal. For the multiplicative subset $S = A \smallsetminus \mathfrak{p}$ we denote $S^{-1}A$ by $A_{\mathfrak{p}}$, called the localisation of A at $\mathfrak{p}$. Show that $A_{\mathfrak{p}}$ is a local ring with maximal ideal $\mathfrak{p}A_{\mathfrak{p}}$ and

$$A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} \simeq \mathrm{Frac}\,(A/\mathfrak{p}),$$

i.e., the fraction field of A at $\mathfrak{p}$.

j) Let M be a finitely generated A module s.t. $M_{\mathfrak{m}} = 0$ for all maximal ideals $\mathfrak{m}$ in A. Show that $M = 0$. Conclude that if $N \subseteq M$ are finitely generated A modules s.t. $M_{\mathfrak{m}} = N_{\mathfrak{m}}$ for all maximal ideals $\mathfrak{m}$ in A, $M = N$.
Hint: Let M be a finitely generated A module s.t. $M_{\mathfrak{m}} = 0$, show that $\mathrm{Ann}(M) \cap A \smallsetminus \mathfrak{m} \neq \emptyset$ where $\mathrm{Ann}(M) := \{a \in A : aM = 0\}$.]

3. The aim of this exercise is to investigate localisation at primes of Dedekind domains. To us, the crucial insight from this exercise is that the localisation of a Dedekind ring at a non-zero prime is a principal ideal domain.

   Let us start by proving that integrality is a local property:

   a) Let $A \subseteq B$ be rings. Prove that B is integral over A iff $B_{\mathfrak{p}}$ is integral over $A_{\mathfrak{p}}$ for all $\mathfrak{p} \lhd A$ prime ideal.

   Now let us assume A is a Dedekind domain and $\mathfrak{p} \lhd A$ is a non-zero prime ideal.

   b) Prove that $A_{\mathfrak{p}}$ is a Dedekind domain.

   c) Prove that the maximal ideal $\mathfrak{p}A_{\mathfrak{p}}$ is a principal ideal. Conclude that $A_{\mathfrak{p}}$ is a P.I.D. Also check that the residue field of $A_{\mathfrak{p}}$,

   $$A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} \simeq A/\mathfrak{p}.$$

   d) Let $\mathfrak{p}A_{\mathfrak{p}} = \pi A_{\mathfrak{p}}$ for $\pi \in A_{\mathfrak{p}}$. Show that every ideal in $A_{\mathfrak{p}}$ satisfies

   $$I = \pi^{v'_{\mathfrak{p}}(I)} A_{\mathfrak{p}}$$

   for a unique integer $v'_{\mathfrak{p}}(I) \in \mathbb{Z}$. Check that this integer is independent of the choice of a generator π of $\mathfrak{p}A_{\mathfrak{p}}$.

   e) For an ideal $I \lhd A$, show that

   $$v'_{\mathfrak{p}}(IA_{\mathfrak{p}}) = v_{\mathfrak{p}}(I)$$

   where the right hand side is the valuation of I at $\mathfrak{p}$ defined in the lecture.

   f) Let A be a noetherian integral domain s.t. for every non-zero prime ideal $\mathfrak{p} \lhd A$, $A_{\mathfrak{p}}$ is a principal ideal domain. Prove that A is a Dedekind domain.

4. Let A be a Dedekind ring and $Q = \mathrm{Frac}(A)$. Let B a domain and $Q \hookrightarrow B$ an embedding. Let $\Lambda \subseteq B$ be a non-zero A-module of finite type and $x \in B$ such that $x.\Lambda \subseteq \Lambda$. Show that $x \in \mathcal{O}_B(A)$.

   Deduce that for any fractional ideal $\mathfrak{f} \subseteq Q$ we have

   $$\{x \in Q \colon x.\mathfrak{f} \subseteq \mathfrak{f}\} = A.$$