Algebraic Number Theory
Manuel Luethi
Federico Viola

Exercise Sheet 13
Spring 2024

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1. Let $d \in \mathbb{Z}$ a square-free integer and let $K = \mathbb{Q}(\sqrt{d})$.
   a) Show that $d \equiv 1, 2, 3 \mod 4$.
   b) Show that
$$\mathcal{O}_K = \begin{cases} \mathbb{Z}[\frac{1+\sqrt{d}}{2}] & \text{if } d \equiv 1 \mod 4, \\ \mathbb{Z}[\sqrt{d}] & \text{otherwise.} \end{cases}$$
2. Let $m, n > 1$ be distinct, squarefree integers. Let $K = \mathbb{Q}(\sqrt{-m}, \sqrt{-n})$ and denote by $\mu_K$ the roots of unity in $K$.
   a) Show that $K/\mathbb{Q}$ is a Galois extension of degree 4 and show that
$$\mathrm{Gal}(K/\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$
   b) Show that $K$ contains exactly three pairwise non-isomorphic proper subfields. List all of them and deduce that $K$ contains a unique real quadratic subfield $L_{\mathbb{R}}$.
   c) Let $L \subseteq K$ be a subfield. Show that $\mathcal{O}_L^\times = L \cap \mathcal{O}_K^\times$.
   d) Let $\sigma \in \mathrm{Gal}(K/\mathbb{Q})$ be the unique non-trivial element fixing $L_{\mathbb{R}} \subseteq K$. Show that the map
$$\phi\colon \mathcal{O}_K^\times \to \mu_K, \quad \varepsilon \longmapsto \frac{\varepsilon}{\sigma(\varepsilon)}$$
   is a well-defined group homomorphism.
   *Hint:* Show that an element of $\mathcal{O}_K^\times$ is a root of unity if and only if the absolute values of all conjugates equal 1. Then apply an explicit description of $\sigma$ in terms of the elements of $\mathrm{Gal}(\mathbb{Q}(\sqrt{-m})/\mathbb{Q})$ and $\mathrm{Gal}(\mathbb{Q}(\sqrt{-n})/\mathbb{Q})$.
   e) Show that $[\mathcal{O}_K^\times \colon \mu_K \mathcal{O}_L^\times] \leqslant 2$.
3. Let $K/\mathbb{Q}$ a totally real number field. Let $d = [K \colon \mathbb{Q}]$. We call an element $x \in K$ totally positive if
$$\forall \sigma \in \mathrm{Hom}_{\mathbb{Q}}(K, \mathbb{R}) \quad \sigma x > 0.$$
   a) Let $\mathcal{F}_K$ denote the group of fractional ideals in $K$ and let $\mathcal{P}_K^+$ be the principal fractional ideals generated by totally positive elements. Prove that $\mathcal{P}_K^+ < \mathcal{F}_K$ is a subgroup and show that
$$\mathrm{Cl}^+(K) = \mathcal{F}_K/\mathcal{P}_K^+$$
   is a finite group.
   b) Let
$$\mathcal{O}_{K}{}_+^\times = \{z \in \mathcal{O}_K^\times \colon z \text{ is totally positive}\}.$$
   Prove that $\mathcal{O}_{K}{}_+^\times \cong \mathbb{Z}^{d-1}$.
   c) Suppose that $\Sigma \subsetneq \mathrm{Hom}_{\mathbb{Q}}(K, \mathbb{R})$ is a proper, non-empty subset. Show that there exists $\varepsilon \in \mathcal{O}_K^\times$ totally positive such that for all $\sigma \in \mathrm{Hom}_{\mathbb{Q}}(K, \mathbb{R})$ we have
$$\sigma\varepsilon \in ]0, 1[ \iff \sigma \in \Sigma.$$
4. a) (Abel's summation formula) Prove the following statement. Given a sequence $(a_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$, for $\xi > 0$ we set $A(\xi) = \sum_{n \leqslant \xi} a_n$. Let $0 < y < x < \infty$ and

$\psi \in \mathrm{C}^1([y, x])$. Then

$$\sum_{y<n<x} a_n \psi(n) = A(x)\psi(x) - A(y)\psi(y) - \int_y^x A(\xi)\psi'(\xi)\mathrm{d}\xi.$$

b) Prove that Riemann's $\zeta$-function

$$\zeta(s) = \sum_{n\in\mathbb{N}} \frac{1}{n^s} \quad (\mathrm{Re}(s) > 1)$$

admits a meromorphic continuation to $\{s\colon \mathrm{Re}(s) > 0\}$ with unique simple pole at $s = 1$ and residue

$$\mathrm{Res}_{s=1}\zeta = 1.$$

c) Let $K/\mathbb{Q}$ be a number field of degree $d$. For all $n \in \mathbb{N}$, let

$$r_K(n) = |\{\mathfrak{a} \lhd \mathcal{O}_K\colon \mathrm{Nr}(\mathfrak{a}) = n\}|.$$

Show that the Dedekind $\zeta$-function $\zeta_K$ has a meromorphic continuation to $\{s\colon \mathrm{Re}(s) > 1 - \frac{1}{d}\}$ with unique simple pole at $s = 1$ and residue

$$\mathrm{Res}_{s=1}\zeta_K = \frac{2^{r_1}(2\pi)^{r_2} h(\mathcal{O}_K)\mathrm{reg}(\mathcal{O}_K)}{\omega_K\sqrt{\mathrm{disc}(\mathcal{O}_K)}},$$

where $\omega_K$ is the number of roots of unity in $K$.