1. Let $K/\mathbb{Q}$ be a number field of degree d with ring of integers $\mathcal{O}_K$. In this exercise we want to find an upper bound for the quantity
$$\#\{\mathfrak{a} \lhd \mathcal{O}_K \colon \mathrm{Nr}(\mathfrak{a}) \leqslant X\}.$$
   a) Let $r_K \colon \mathbb{N} \to \mathbb{C}$ be the arithmetic function given by
   $$r_K(n) = \#\{\mathfrak{a} \lhd \mathcal{O}_K \colon \mathrm{Nr}(\mathfrak{a}) = n\}.$$
   Prove that r_K is multiplicative, i.e., that
   $$r_K(n_1 n_2) = r_K(n_1) r_K(n_2) \quad \text{for} \quad (n_1, n_2) = 1.$$
   b) Show that, for any prime p and any positive integer ℓ, we have the bound
   $$r_K(p^\ell) \leqslant (\ell+1)^d.$$
   c) Let $\varepsilon > 0$. Show that there exists a constant $C_\varepsilon > 0$ such that
   $$r_K(n) \leqslant C_\varepsilon n^\varepsilon \quad \text{for all} \quad n \in \mathbb{N}.$$
   Hint: Let $\mathcal{P}_\varepsilon$ be the set defined by
   $$\mathcal{P}_\varepsilon = \{p \text{ prime} \colon p \leqslant e^{\frac{d}{\varepsilon}}\}.$$
   Show that
   $$\frac{(\ell+1)^d}{p^{\varepsilon \ell}} \leqslant 1 \qquad \text{for all} \quad p \notin \mathcal{P}_\varepsilon \quad \text{and} \quad \ell \in \mathbb{N}.$$
   Then define a number M_ε by
   $$M_\varepsilon = \max_{\lambda \in [0,\infty)} \frac{(\lambda+1)^d}{2^{\varepsilon\lambda}}$$
   and show that
   $$\frac{(\ell+1)^d}{p^{\varepsilon \ell}} \leqslant M_\varepsilon \qquad \text{for all} \quad p \in \mathcal{P}_\varepsilon \quad \text{and} \quad \ell \in \mathbb{N}.$$
   d) Conclude that for any $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$such that
   $$\#\{\mathfrak{a} \text{ an ideal of } \mathcal{O}_K \colon \mathrm{N}(\mathfrak{a}) \leqslant X\} \leqslant C_\varepsilon X^{1+\varepsilon}.$$
2. Let $K/\mathbb{Q}$ a number field of degree d. Let $r_1 = |\mathrm{Hom}_{\mathbb{Q}}(K, \mathbb{R})|$ and $r_2 = (d - r_1)/2$. Throughout this exercise, we consider a fixed choice
$$\Sigma = (\sigma_1, \ldots, \sigma_{r_1+r_2}) \in \mathrm{Hom}_{\mathbb{Q}}(K, \mathbb{C})^{r_1+r_2}$$
of an ordered, complete set of representatives of $\mathrm{Hom}_{\mathbb{Q}}(K, \mathbb{C})/\mathrm{Gal}(\mathbb{C}, \mathbb{R})$ such that
$$\mathrm{Hom}_{\mathbb{Q}}(K, \mathbb{R}) = \{\sigma_1, \ldots, \sigma_{r_1}\}.$$
We define
$$\sigma_\infty \colon K \to \mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2}, \quad \sigma_\infty(z) = (\sigma_1 z, \ldots, \sigma_{r_1+r_2} z).$$
   a) Let $\mathfrak{f} \subseteq K$ be a non-zero fractional ideal (not necessarily proper). Prove that
   $$\mathrm{covol}\big(\sigma_\infty(\mathfrak{f})\big) = 2^{-r_2} |\mathrm{disc}(\mathcal{O}_K)|^{1/2} \mathrm{Nr}_{K/\mathbb{Q}}(\mathfrak{f}).$$

b) Prove that

$$\mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2} \cong K \otimes_{\mathbb{Q}} \mathbb{R}$$

as $\mathbb{R}$-algebras.

3. Let $K/\mathbb{Q}$ a number field. Show that

$$|\mathrm{disc}(K)| \geqslant \frac{\pi}{3}\left(\frac{3\pi}{4}\right)^{[K:\,\mathbb{Q}]-1}$$

and deduce that for any number field $K/\mathbb{Q}$ not equal to $\mathbb{Q}$, there is a prime $p \in \mathbb{Z}$ that ramifies in K.
Hint: Set

$$a_d = \left(\frac{d^d}{d!}\left(\frac{\pi}{4}\right)^{\frac{d}{2}}\right)^2 \qquad (d \geqslant 2).$$

Calculate a_2 and prove that $a_{d+1} \geqslant a_d$. Then use that $\mathcal{O}_K$ contains a non-zero ideal $\mathfrak{a}$ such that

$$\mathrm{Nr}(\mathfrak{a})^2 \leqslant \left(\frac{d!}{d^d}\left(\frac{4}{\pi}\right)^{r_2}\right)^2 |\mathrm{disc}(K)|.$$

4. Show that for every $d \geqslant 2$, for every $\varepsilon > 0$, there exists a constant $C_\varepsilon \geqslant 0$ such that the following is true. Let K be a number field of degree d, then

$$|\mathrm{Cl}(K)| \leqslant C_\varepsilon |\mathrm{disc}(K)|^{\frac{1}{2}+\varepsilon}.$$

Hint: Use Exercise 1d.

5. Let $K/\mathbb{Q}$ a number field of degree d, $r_1 = |\mathrm{Hom}_{\mathbb{Q}}(K,\mathbb{C})|$, and $r_2 = (d-r_1)/2$. Show that every ideal class contains an integral ideal containing an integer a satisfying

$$1 \leqslant a \leqslant \left(\frac{4}{\pi}\right)^{r_2} \frac{d!}{d^d} |\,\mathrm{disc}(K)|^{\frac{1}{2}} =: C.$$

Deduce that $\mathrm{Cl}(K)$ is generated by classes of elements of

$$\bigsqcup_{1 \leqslant p \leqslant C} \mathrm{Spec}_p(\mathcal{O}_K),$$

where the union runs over rational primes.
Hint: Let $\mathfrak{a} \lhd \mathcal{O}_K$ be an appropriately chosen non-zero ideal and consider the generator a of $\mathbb{Z} \cap \mathfrak{a}$.

6. Determine the class group of $\mathbb{Q}(\sqrt{-19})$.

7. Determine the class group of $\mathbb{Q}(\sqrt{-5})$.
Hint: Is $(2, 1+\sqrt{-5}) \subseteq \mathbb{Z}[\sqrt{-5}] = \mathcal{O}_K$ a principal ideal?

8. a) Use SageMATH to perform the following computation (using whatever built-in function you can find). Given a fundamental discriminant $D \in \mathbb{Z}$, i.e., a number $D \neq 0, 1$ such that
   - either $D \equiv 1 \mod (4)$ and square-free,
   - or $D \equiv 0 \mod (4)$ such that $D/4$ is square-free and $D/4 \equiv 2, 3 \mod (4)$,

   calculate the class number $h(D)$ of the quadratic number field of discriminant D.

   b) Guess the asymptotics for $h(D)$ as $D \to -\infty$ among fundamental discriminants.

   c) Guess the asymptotics of the relative number of positive fundamental discriminants such that $h(D) = 1$ as $D \to \infty$ among fundamental discriminants. What do you obtain if you restrict to prime fundamental discriminants?