Serie 9
Optimal transport, Fall semester
EPFL, Mathematics section, Dr. Xavier Fernández-Real

Exercise 9.1 (Wasserstein and L^p-distances are not comparable)**.** Let $p \in [1, \infty)$. Give an example of two sequences of compactly supported nonnegative functions $f_n, g_n \in L^p(\mathbb{R}^d)$ with $\int f_n = \int g_n = 1$ for which, calling $\mu_n = f_n \mathscr{L}^d, \nu_n = g_n \mathscr{L}^d$ we have (give an example for each of the two scenarios):

i) $W_p(\mu_n, \nu_n) \to 0$ and $\|f_n - g_n\|_{L^p} \geq \epsilon > 0$.

ii) $W_p(\mu_n, \nu_n) \geq \epsilon > 0$ and $\|f_n - g_n\|_{L^p} \to 0$.

Exercise 9.2 (Convergence of p-Wasserstein distance as $p \downarrow 1$)**.** Let $\mu, \nu \in \mathcal{P}(\mathbb{R}^d)$ be a pair of probability measures. Show that if μ and ν are supported on a compact set, then

$$\lim_{p \downarrow 1} W_p(\mu, \nu) = W_1(\mu, \nu)\,.$$

Show a counterexample to the previous statement if we drop the assumption that μ and ν are supported on a compact set.

Hint: For the counterexample, set $\mu = \delta_0$ and find a measure ν such that $W_p(\mu, \nu) = \infty$ if $p > 1$ and $W_1(\mu, \nu)$ is finite.

Exercise 9.3 (Convergence of p-Wasserstein distance as $p \uparrow \infty$)**.** Let $\mu, \nu \in \mathcal{P}(\mathbb{R}^d)$ be two compactly supported probability measures. The ∞-Wasserstein distance between μ and ν is defined as

$$W_\infty(\mu, \nu) := \inf\left\{ \|x - y\|_{L^\infty(\mathbb{R}^d \times \mathbb{R}^d, \gamma)} : \gamma \in \Gamma(\mu, \nu) \right\}.$$

i) Prove that $W_p(\mu, \nu) \uparrow W_\infty(\mu, \nu)$ as $p \uparrow \infty$. Deduce that W_∞ defines a distance on $\mathcal{P}_\infty(\mathbb{R}^d) := \{\mu \in \mathcal{P}(\mathbb{R}^d) : \ \mu \text{ has compact support}\}$.

ii) Give an example of $\mu_n, \mu \in \mathcal{P}(\mathbb{R}^d)$ compactly supported in a common compact set for which

$$\begin{cases} W_p(\mu_n, \mu) \to 0 & \text{for every } p \in [1, \infty), \\ W_\infty(\mu_n, \mu) \geq \epsilon > 0 & \text{for every } n. \end{cases}$$

Exercise 9.4 (💣)**.** For every $p \in [1, \infty)$, show that $(\mathcal{P}_p(\mathbb{R}^d), W_p)$ is a Polish space (separable and complete).

Hints: You can solve the exercise via the following steps:

i) For the separability, approximate each $\mu \in \mathcal{P}_p(\mathbb{R}^d)$ with finite sums of dirac deltas in rational points and with rational coefficients.

ii) To prove completeness, take a Cauchy sequence $\{\mu_n\}_{n\geq 1} \subset \mathcal{P}_p(\mathbb{R}^d)$ and argue as follows:

- For every $k \geq 1$ take an optimal $\gamma_k \in \Gamma(\mu_k, \mu_{k+1})$. Use the disintegration Theorem to build a sequence of measures $\pi_n \in \mathcal{P}((\mathbb{R}^d)^n)$ with the following properties:

$$\begin{cases} p^{1,\dots,n-1}_{\#}\pi_n = \pi_{n-1} & \text{for every } n \geq 2, \\ p^{k,k+1}_{\#}\pi_n = \gamma_k & \text{for every } 1 \leq k < n. \end{cases}$$

  Here $p^{i,\dots,j}$ denotes the projection on the variables from i to j.

- Use Kolmogorov's extension Theorem to find $\pi_\infty \in \mathcal{P}((\mathbb{R}^d)^{\mathbb{N}})$ such that

$$p^{1,\dots,n}_{\#}\pi_\infty = \pi_n \quad \text{for every } n \geq 1.$$

- Observe that the L^p-space

$$\mathcal{X} := L^p((\mathbb{R}^d)^{\mathbb{N}}, \pi_\infty)$$

  is complete. Assuming without loss of generality that $\sum_n W_p(\mu_n, \mu_{n+1}) < \infty$, prove that the coordinate functions $p^n : (\mathbb{R}^d)^{\mathbb{N}} \to \mathbb{R}$ form a Cauchy sequence in $\mathcal{X}$, and deduce that $p^n \to \bar{p}$ in $\mathcal{X}$.

- Conclude that $\mu_n \to \bar{\mu}$ in $(\mathcal{P}_p(\mathbb{R}^d), W_p)$, where $\bar{\mu} := \bar{p}_{\#}\pi_\infty$.