Serie 8

Optimal transport, Fall semester

EPFL, Mathematics section, Dr. Xavier Fernández-Real

Exercise 8.1. Assume $\mu, \nu \in \mathcal{P}(\mathbb{R})$ and μ be non atomic. We define

$$T_{mon}(x) := F_\nu^{[-1]} \circ F_\mu(x),$$

where F denotes the cumulative distribution of a probability measure and $F^{[-1]}$ its pseudo-inverse. Show that the nondecreasing map $T_{mon} : \mathbb{R} \to \mathbb{R}$ constructed in class is optimal with respect to the quadratic cost.

The rest of the exercise sheet is devoted to the disintegration theorem. The next exercise shows the intuition behind the disintegration, whereas the following ones are increasingly general statements of the theorem. The main difficulty of the proof is contained entirely in Exercise 8.3.

Exercise 8.2. Let $\mu \in \mathscr{M}(\mathbb{R}^2)$ be a finite measure on $\mathbb{R}^2$ that is absolutely continuous with respect to the Lebesgue measure with density $\rho : \mathbb{R}^2 \to \mathbb{R}$. Let $\nu \in \mathscr{M}(\mathbb{R})$ be the measure with density $\eta(x) = \int_{\mathbb{R}} \rho(x, y)dy$. For any $x \in \mathbb{R}$ such that $\eta(x) \neq 0$, let μ_x be the measure with density $\rho_x(y) = \frac{\rho(x,y)}{\eta(x)}$. If $\eta(x) = 0$, then simply set $\mu_x = 0$.

Show that for any $g \in L^1(\mu)$ it holds

$$\int_{\mathbb{R}^2} g(x, y)d\mu(x, y) = \int_{\mathbb{R}} \int_{\mathbb{R}} g(x, y)d\mu_x(y)d\nu(x).$$

Exercise 8.3 (Disintegration for product of compact spaces)**.** Let X, Y be two compact spaces and let $\mu \in \mathscr{M}(X \times Y)$ be a finite measure on the product $X \times Y$. Let us denote $\nu = (\pi_1)_\#\mu$ where $\pi_1 : X \times Y \to X$ is the projection on the first coordinate. Prove that there exists a family of probabilities $(\mu_x)_{x \in X} \subseteq \mathcal{P}(Y)$ such that for any $g \in L^1(\mu)$ it holds

$$\int_{X \times Y} g(x, y)d\mu(x, y) = \int_X \int_Y g(x, y)d\mu_x(y)d\nu(x). \tag{1}$$

Do it by following the next steps:

(i) Given $\psi \in C^0(Y)$, consider the map $A_\psi : L^1(X, \nu) \to \mathbb{R}$ given by the formula $A_\psi(\phi) := \int_{X \times Y} \phi(x)\psi(y)d\mu(x, y)$. Prove that the said map is linear continuous and therefore A_ψ can be represented by a function in $L^\infty(X, \nu)$. As an abuse of notation, we denote by $A_\psi(x) \in L^\infty(X, \nu)$ such function, so that the previous map is $\phi \mapsto \int_X \phi(x)A_\psi(x)d\nu(x)$.

(ii) Fix a countable dense subset $S \subseteq C^0(Y)$. Prove that for ν-almost every $x \in X$ the map $\mu_x : S \to \mathbb{R}$ given by $\mu_x(\psi) = A_\psi(x)$ is linear continuous and therefore $\mu_x \in \mathcal{P}(Y)$. Assume that the said family $(\mu_x)_{x \in X}$ satisfies that for any Borel set $E \subset Y$, the map $x \mapsto \mu_x(E)$ is ν-measurable (this is necessary to give a meaning to the integral in the statement).

(iii) Show that the desired statement holds when $g \in L^1(X,\nu) \times S$. Show that this implies that it holds also when $g \in L^1(X,\nu) \times C^0(Y)$. Finally show that this implies it holds also for any $g \in L^1(\mu)$.

Exercise 8.4 (). [Disintegration for product of Polish spaces] Show the statement of the previous exercise when X and Y are Polish spaces, i.e. they are complete and separable.

Hint: Use Prokhorov's theorem (and Lemma 2.1.9) to find a suitable exhaustion in compact sets that allows to apply the previous exercise.

Exercise 8.5 (Disintegration for fibers of a map)**.** Let X, Y be two Polish spaces, let $f : Y \to X$ be a Borel map and let $\mu \in \mathscr{M}(Y)$ be a finite measure on Y. Let us denote $\nu := f_\# \mu$. Show that there exists a family of probabilities $(\mu_x)_{x \in X} \subseteq \mathcal{P}(Y)$ such that:

(i) For ν-almost every $x \in X$ the measure μ_x is supported on the fiber $f^{-1}(x)$.

(ii) For any $g \in L^1(\mu)$ it holds

$$\int_Y g(y) d\mu(y) = \int_X \int_{f^{-1}(x)} g(y) d\mu_x(y) d\nu(x).$$

Hint: Apply the previous exercise on the measure $(f \times \mathrm{id})_\# \mu$.