

Serie 4
Optimal transport, Fall semester
EPFL, Mathematics section, Dr. Xavier Fernández-Real

Given a function $f : \mathbb{R}^d \rightarrow \mathbb{R} \cup \{-\infty, \infty\}$ we define the convex conjugate f^* as

$$f^*(y) = \sup_{x \in \mathbb{R}^d} (x \cdot y - f(x)).$$

When $f : \mathbb{R}^d \rightarrow \mathbb{R} \cup \{+\infty\}$ is convex, f^* is also known as Legendre transform of f . Notice that, at least informally, if we assume that f is differentiable and that the supremum in the right-hand side is realized at a point $\bar{x}$, then $y = \nabla f(\bar{x})$.

Exercise 4.1. Given two functions $f, g : \mathbb{R}^d \rightarrow \mathbb{R} \cup \{+\infty\}$ such that $f, g \not\equiv +\infty$. Show the following:

- (i) f^* and g^* are convex functions.
- (ii) If $f \leq g$, then $g^* \leq f^*$.

Exercise 4.2. Compute the convex conjugate of

- (i) $f(x) = \frac{1}{2}\langle x, x \rangle$ for $x \in X$, $X = \mathbb{R}^d$;
- (ii) $f(x) = \langle x, x_0 \rangle$, for $x \in X$, where $x_0 \in X$ is a fixed point, $X = \mathbb{R}^d$;
- (iii) a function f defined by $f(x_0) = 0$ and for $x \in X$, $x \neq x_0$, $f(x) = +\infty$, where $x_0 \in X$ is a fixed point, $X = \mathbb{R}^d$;
- (iv) $f(x) = \frac{1}{p}|x|^p$ if $1 < p < \infty$ and $X = \mathbb{R}$.

Exercise 4.3. Let $f : \mathbb{R}^d \rightarrow \mathbb{R} \cup \{+\infty\}$ be a convex lower semicontinuous function such that $f \not\equiv +\infty$. Prove that $(f^*)^* = f$.

Hint: Prove the two inequalities separately, $f \geq (f^*)^*$ and $f \leq (f^*)^*$. For the latter, use point (ii) and (iii) of Exercise 4.2.

Exercise 4.4. Given a function $f : \mathbb{R}^d \rightarrow \mathbb{R} \cup \{+\infty\}$, prove that

- (i) $y \in \partial f(x)$ if and only if $f(x) + f^*(y) = \langle x, y \rangle$;
- (ii) If f is convex, lower semicontinuous, and $f \not\equiv +\infty$, then $y \in \partial f(x) \iff x \in \partial f^*(y)$.

Exercise 4.5 (★). Consider a strictly convex C^1 function $f : \mathbb{R}^d \rightarrow \mathbb{R}$ such that

$$\lim_{|x| \rightarrow \infty} \frac{f(x)}{|x|} = +\infty.$$

Prove that $\nabla f : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a bijection and $f(x) + f^*(y) = \langle x, y \rangle$ if and only if $\nabla f(x) = y$.