Serie 3

Optimal transport, Fall semester

EPFL, Mathematics section, Dr. Xavier Fernández-Real

Exercise 3.1. Let $f : \mathbb{R}^d \to \mathbb{R}$ be a nonnegative lower semicontinuous function. Show that:

(i) f admits a minimizer in every compact set $K \subset \mathbb{R}^n$.

(ii) f can be approximated from below monotonically (namely, $f_\lambda(x) \uparrow f(x)$ for every $x \in \mathbb{R}^d$) by a sequence of functions f_λ as $\lambda \to \infty$, where f_λ is λ-Lipschitz.

(iii) for every sequence of measures $\mu_n \rightharpoonup \mu$ narrowly,

$$\liminf_{n\to\infty} \int_{\mathbb{R}^d} f\, d\mu_n \geq \int_{\mathbb{R}^d} f\, d\mu.$$

Hint: For (ii), define $f_\lambda(x) = \inf_{y\in\mathbb{R}^n} \{f(y) + \lambda|x-y|\}$.

Exercise 3.2. The support of a nonnegative measure $\mu \in \mathcal{M}_+(\mathbb{R}^n)$ is defined as the smallest closed set on which μ is concentrated, i.e.

$$spt(\mu) := \bigcap \{C \subset \mathbb{R}^n \text{ closed } : \mu(\mathbb{R}^n \setminus C) = 0\}.$$

Let us take a sequence of nonnegative measures $\mu_j \in \mathcal{M}_+(\mathbb{R}^n)$ such that $\mu_j \overset{*}{\rightharpoonup} \mu$. Prove the following fact: for every $x \in spt(\mu)$, there exists a sequence of points $x_j \in spt(\mu_j)$ such that $x_j \to x$.

Exercise 3.3 (Characterizations of weak-$*$ convergence)**.** Let $\mu_j, \mu \in \mathscr{P}(\mathbb{R}^n)$ be probability measures in $\mathbb{R}^n$.

i) Show that $\mu_j \rightharpoonup \mu$ narrowly if and only if one of the following properties hold:

a) For every open set $A \subset \mathbb{R}^n$:

$$\liminf_{j\to\infty} \mu_j(A) \geq \mu(A).$$

b) For every closed set $C \subset \mathbb{R}^n$:

$$\limsup_{j\to\infty} \mu_j(C) \leq \mu(C).$$

c) For every set $E \subset \mathbb{R}^n$ such that $\mu(\partial E) = 0$:

$$\lim_{j\to\infty} \mu_j(E) = \mu(E).$$

ii) Give an example of a sequence of probability measures $\mu_j \in \mathscr{P}(\mathbb{R}^n)$ such that $\mu_j \overset{*}{\rightharpoonup} \mu$ for some measure $\mu \in \mathcal{M}_+(\mathbb{R}^n)$ and an open set A such that

$$\liminf_{j\to\infty} \mu_j(A) > \mu(A).$$

Hint: For one implication in (i), use Exercise 3.1. For the other, use the layer-cake formula

$$\int \varphi d\mu = \int_0^\infty \mu(\{\varphi > t\})dt \quad \text{for every } \varphi \in C_b(\mathbb{R}^n),\ \varphi \geq 0.$$

Exercise 3.4. Let $\{\mu_n\}_{n\in\mathbb{N}} \subset \mathscr{P}(\mathbb{R})$ be a sequence of probability measures with $\mu_n \rightharpoonup \mu$ narrowly. Define $F_n(x) := \mu_n((-\infty, x])$, $F(x) := \mu((-\infty, x])$.

(i) Prove that μ is a probability measure.

(ii) Prove that

$$\limsup_n F_n(x) \leq F(x) \qquad \text{for every } x \in \mathbb{R}.$$

(iii) Prove that

$$\lim_n F_n(x) = F(x) \quad \text{for every } x \in \mathbb{R} \text{ at which } F \text{ is continuous.}$$

(iv) Give an example of a sequence of measures $\mu_n \rightharpoonup \mu$ narrowly and an $x \in \mathbb{R}$ for which

$$\limsup_n F_n(x) < F(x).$$

Exercise 3.5.

(i) Find a sequence of functions $f_n : [0,1] \to [0,1]$ such that $(f_n)_\#(\mathscr{L}^1 \llcorner [0,1]) = (\mathscr{L}^1 \llcorner [0,1])$ but f_n weakly converge to $1/2$.

(ii) What is the weak limit of $(\mathrm{id}, f_n)_\# \mathscr{L}^1 \llcorner [0,1]$?

(iii) (💣) Can these functions be taken C^1?

Hint: For (i), use piecewise affine oscillating functions.

Exercise 3.6 (💣). Let $\mu \in \mathscr{P}(\mathbb{R}^n)$ be a probability measure. We say that a sequence of borel functions $T_j : \mathbb{R}^n \to \mathbb{R}^n$ converge in μ-measure to $T : \mathbb{R}^n \to \mathbb{R}^n$ if[1]

$$\lim_{j\to\infty} \mu\left(\{x \in \mathbb{R}^n : |T_j(x) - T(x)| > \epsilon\}\right) = 0 \quad \text{for every } \epsilon > 0.$$

Denoting by $\pi_j := (id, T_j)_\# \mu$, $\pi := (id, T)_\# \mu \in \mathscr{P}(\mathbb{R}^n \times \mathbb{R}^n)$, prove the following equivalence:

$$\pi_j \overset{*}{\rightharpoonup} \pi \qquad \Longleftrightarrow \qquad T_j \text{ converges to } T \text{ in } \mu\text{-measure}$$

[1] By standard measure theory arguments one can easily prove that whenever T_j converges to T in μ-measure, there is a subsequence $j_k \nearrow \infty$ such that T_{j_k} converge to T pointwise μ-almost everywhere.