Serie 2

Optimal transport, Fall semester

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Exercise 2.1. Determine the measure $T_\# \mu$ for the map $T = \nabla \varphi : \mathbb{R}^2 \to \mathbb{R}^2$, where the function $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ is given by $\varphi(x_1, x_2) = \frac{1}{3}(x_1^3 + x_2^3)$, for the following choices of μ:

(i) $\mu = \delta_{(0,0)} + \delta_{(1,2)}$;

(ii) $\mu = \mathscr{L}^2 \llcorner [0,1]^2$.

Consider the Monge problem with the cost function $c(x,y) = \frac{|x-y|^2}{2}$. Is the map T optimal from μ to $T_\# \mu$, for μ as in (i)?

Exercise 2.2. Let us introduce the following notation for the Monge problem:

$$MP(\mu, \nu, c) := \inf \left\{ \int_X c(x, T(x)) d\mu(x) : T_\# \mu = \nu \right\}.$$

(i) Give a cost function $c : \mathbb{R}^2 \times \mathbb{R}^2 \to [0, \infty)$ of the form $c(x,y) = f(|x-y|)$ for some Lipschitz function $f : [0,\infty) \to [0,\infty)$ and a measure μ for which the identity is not an optimal map for the problem $MP(\mu, \mu, c)$. More precisely, give a cost function c as above, a measure μ and a map T such that $T_\# \mu = \mu$ and

$$\int_{\mathbb{R}^2} c(x, T(x)) d\mu(x) < \int_{\mathbb{R}^2} c(x, x) d\mu(x).$$

(ii) Let $c : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$ be a measurable cost function such that $c(x,y) = f(|x-y|)$ for a measurable $f : [0,\infty) \to [0,\infty)$. What is a necessary and sufficient condition on the cost function c in such a way that the identity is an optimal map for $MP(c, \mu, \mu)$ for any $\mu \in \mathscr{P}(\mathbb{R}^n)$?

Exercise 2.3. Find the unique monotone map $T : [0,1] \to [0,\infty)$ such that $T_\# \mu = \nu$, where the measures $\mu, \nu \in \mathscr{P}(\mathbb{R})$ are given by $\mu = \mathscr{L}^1 \llcorner [0,1]$ and $\nu = e^{-x} \mathscr{L}^1 \llcorner [0, \infty)$.

The next three exercises are devoted to the derivation of some properties of the Knothe transport map. Given $f, g : \mathbb{R}^2 \to \mathbb{R}$ two positive functions with integral 1, we define

$$F(x_1) = \int_{\mathbb{R}} f(x_1, x_2)\, dx_2, \qquad G(x_1) = \int_{\mathbb{R}} g(x_1, x_2)\, dx_2.$$

Let $T_1 : \mathbb{R} \to \mathbb{R}$ be the monotone map which sends $F(x_1)\, dx_1$ to $G(x_1)\, dx_1$, namely $(T_1)_\#(F(x_1)\, dx_1) = G(x_1)\, dx_1$. For every $x_1 \in \mathbb{R}$, let $T_2(x_1, \cdot)$ be the monotone map which sends $f(x_1, x_2)/F(x_1)\, dx_2$ to $g(T_1(x_1), x_2)/G(T_1(x_1))\, dx_2$. The Knothe's map $T : \mathbb{R}^2 \to \mathbb{R}^2$ is defined as

$$T(x_1, x_2) = (T_1(x_1), T_2(x_1, x_2)).$$

Exercise 2.4. Show that the Knothe's map $T(x_1, x_2) = (T_1(x_1), T_2(x_1, x_2))$ transports the measure $\mu = f(x_1, x_2)\, dx_1 dx_2$ to $\nu = g(x_1, x_2)\, dx_1 dx_2$.

Exercise 2.5. Let T be a Knothe's map from $\mu = \dfrac{\mathbb{1}_E}{|E|}\, dx$ to $\nu = \dfrac{\mathbb{1}_{B_1}}{|B_1|}\, dy$, where $B_1 \subseteq \mathbb{R}^2$ is the unit ball and $E \subseteq \mathbb{R}^2$ a bounded open set with smooth boundary. Assuming that T is smooth, show that

(i) For any $x \in E$, it holds $|T(x)| \leq 1$.

(ii) $\det \nabla T = \dfrac{|B_1|}{|E|}$ in E.

(iii) $\operatorname{div} T \geq 2(\det \nabla T)^{\frac{1}{2}}$.

Hint: For (ii) and (iii), notice that the Jacobian ∇T of a Knothe's map T is upper triangular and all values on the diagonal are non-negative.

Exercise 2.6 (Isoperimetric inequality in $\mathbb{R}^2$)**.** Let $E \subset \mathbb{R}^2$ be a bounded open set with smooth boundary. Show that

$$\mathrm{Length}(\partial E) \geq 2|B_1|^{\frac{1}{2}}|E|^{\frac{1}{2}}.$$

Remark 2.1. The definition of the Knothe's map as well as the proof of the three previous exercises can be carried out in the same way in a general dimension d without many new ideas but at the price of a heavier notation. The student is invited to try himself to generalize to the d-dimensional case the definition of Knothe's map, its properties, and its use in the proof of the Isoperimetric inequality.