Serie 13

Optimal transport, Fall semester

EPFL, Mathematics section, Dr. Xavier Fernández-Real

Exercise 13.1. Let $\Omega \subseteq \mathbb{R}^d$ be an open bounded set, and $V_1 : \Omega \to \mathbb{R}$, $V_2 : \Omega \to \mathbb{R}$ be functions which are lower semicontinuous and bounded from below. Show that the functionals

$$\mathbb{V}_1(\mu) = \int_\Omega V_1 \, d\mu \qquad \mathbb{V}_2(\mu) = \int_\Omega V_2(x,y) \, d\mu(x) \, d\mu(y)$$

are lower semicontinuous with respect to W_2-convergence.

Exercise 13.2. Show that the functional $\mathscr{F}$ given by

$$\mathscr{F}(\rho) := \int_{\mathbb{R}^d} (\rho + |x|^2)\rho dx$$

is W_2-convex, and compute the evolution equation of its Wasserstein gradient flow.

Exercise 13.3. Let $\mu := \frac{1}{\pi}\chi_{B(0,1)}\mathcal{L}^2$ be the uniform probability measure on $B(0,1) \subset \mathbb{R}^2$, and let $p_1 := (1,0), p_2 := (2,0) \in \mathbb{R}^2$. Describe the optimal transport map between μ and $\frac{1}{2}(\delta_{p_1} + \delta_{p_2})$ in the following two cases:

(i) when the cost is the quadratic cost $\frac{1}{2}|x-y|^2$;

(ii) when the cost is the linear cost $|x-y|$. In this case, there is no need to write the full explicit map, a general form is already enough.

Exercise 13.4. Consider n red points $P_1, \ldots, P_n$ and n blue points $Q_1, \ldots, Q_n$ on the plane. Assume that these $2n$ points are distinct and there are no 3 collinear points.

Show that it is possible to connect each red point to a distinct blue point with a segment in such a way that these segments do not intersect each other. Namely, there exists a permutation $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ such that the segment $\overline{P_i Q_{\sigma(i)}}$ does not intersect the segment $\overline{P_j Q_{\sigma(j)}}$ for any $i \neq j$.