

Serie 10

Optimal transport, Fall semester

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Definition 1 (Middle point). Given two probability measures $\mu_0, \mu_1 \in \mathcal{P}(\mathbb{R}^d)$, let $\mathcal{C}(\mu_0, \mu_1)$ be the infimum of the Kantorovich problem with respect to the quadratic cost

$$\mathcal{C}(\mu_0, \mu_1) := \inf_{\gamma \in \Gamma(\mu_0, \mu_1)} \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|x - y|^2}{2} d\gamma(x, y).$$

Let $\mu_0, \mu_1 \in \mathcal{P}(\mathbb{R}^d)$ be two probability measures with compact support. A probability measure $\mu_{\frac{1}{2}}$ is a middle point of μ_0 and μ_1 if $\mathcal{C}(\mu_0, \mu_{\frac{1}{2}}) = \mathcal{C}(\mu_1, \mu_{\frac{1}{2}}) = \frac{1}{4}\mathcal{C}(\mu_0, \mu_1)$.

Exercise 10.1. For any $\rho \in \mathcal{P}(\mathbb{R}^d)$, it holds

$$\mathcal{C}(\mu_0, \rho) + \mathcal{C}(\rho, \mu_1) \geq \frac{1}{2}\mathcal{C}(\mu_0, \mu_1).$$

Moreover, if equality holds, then there is an optimal plan $\gamma \in \Gamma_{opt}(\mu_0, \mu_1)$ such that $(\frac{x+z}{2})_{\#}\gamma = \rho$ (here, x, z denote the first and second coordinate of $\mathbb{R}^d \times \mathbb{R}^d$).

Hint: See the proof of Theorem 3.1.5. Try to prove the inequality without directly using the triangular inequality for the Wasserstein distance.

Exercise 10.2. Thanks to the previous Exercise, show that a measure $\mu_{\frac{1}{2}}$ is a middle-point if and only if

$$\mu_{\frac{1}{2}} \text{ is a middle-point} \quad \Leftrightarrow \quad \mathcal{C}(\mu_0, \mu_{\frac{1}{2}}) \leq \frac{1}{4}\mathcal{C}(\mu_0, \mu_1) \quad \text{and} \quad \mathcal{C}(\mu_1, \mu_{\frac{1}{2}}) \leq \frac{1}{4}\mathcal{C}(\mu_0, \mu_1). \quad (1)$$

Exercise 10.3. Let $\mu_0, \mu_1 \in \mathcal{P}(\mathbb{R}^d)$ be two probability measures with compact support. A probability measure $\mu_{\frac{1}{2}}$ is a middle point of μ_0 and μ_1 if $\mathcal{C}(\mu_0, \mu_{\frac{1}{2}}) = \mathcal{C}(\mu_1, \mu_{\frac{1}{2}}) = \frac{1}{4}\mathcal{C}(\mu_0, \mu_1)$.

(i) If $\mu_0 = \delta_{p_0}$ and $\mu_1 = \delta_{p_1}$, show that the middle point is unique and $\mu_{\frac{1}{2}} = \delta_{\frac{p_0+p_1}{2}}$.

(ii) Prove that there is always at least one middle point.

(iii) Find two probability measures μ_0, μ_1 such that they have more than one middle point.

(iv) Show that if the optimal transport plan between μ_0 and μ_1 is unique, then there is a unique middle point.

(v) Prove that if $\mu_0, \mu_1 \ll \mathcal{L}^d$, then the middle point is unique and $\mu_{\frac{1}{2}} \ll \mathcal{L}^d$.