Characteristics and Weak solutions

Exercise 1
Consider the partial differential equation

$$\partial_t u(x,t) + x\partial_x u(x,t) = 0.$$

Find curves $x(t)$ along which any solution must be constant. Show that every $(x,t) \in \mathbb{R} \times \mathbb{R}$ lies on such a curve. Find a solution for the initial values $u_0(x) = \cos(x)$.

Solution 1
From the equation, we can tell that

$$\begin{pmatrix} \partial_x u(x,t) \\ \partial_t u(x,t) \end{pmatrix} \cdot \begin{pmatrix} x \\ 1 \end{pmatrix} = 0$$

At every point (x,t), the solution must be constant if we move in direction $(x,1)$. A curve $x(t)$ must therefore obey the ordinary differential equation

$$\partial_t x(t) = x(t), \quad x(0) = x_0.$$

The solution is $x(t) = x_0 \exp(t)$.

Given any (x,t), we see that (x,t) lies on the characteristic starting at $x_0 = x\exp(-t)$.

Since u is constant along each characteristic, it follows that

$$u(x,t) = u\left(x \cdot e^{-t}, 0\right) = u_0\left(x \cdot e^{-t}\right)$$

If the initial values are $u_0(x) = \sin(x)$, we thus find the solution by:

$$u(x,t) = \sin\left(x \cdot e^{-t}\right).$$

Exercise 2
Suppose that u is the density of a quantity over the real line. We assume that its velocity over the real line is given by

$$v(x,t) = u(x,t)^3.$$

Describe the change of the integral of u over a subinterval $[x_1, x_2]$ in terms of the velocity v and u itself. Use the fundamental theorem of calculus to derive a conservation law, and find its characteristics. Discuss the similarity to Burgers' equation, and how this example can be generalized.

Solution 2
We consider the time-dependent integral

$$M(t) = \int_{x_1}^{x_2} u(x,t)dx.$$

The influx/outflux of the endpoint is described by the velocity

$$\partial_t M(t) = \partial_t \int_{x_1}^{x_2} u(x,t)dx = u(x_1,t)^3 u(x_1,t) - u(x_2,t)^3 u(x_2,t) = u(x_1,t)^4 - u(x_2,t)^4.$$

Via the fundamental theorem of calculus,

$$\partial_t \int_{x_1}^{x_2} u(x,t)dx = -\int_{x_1}^{x_2} \partial_x u(x,t)^4 dx.$$

We take the time-derivative under the integral. If we take the intervals infinitesimally small, then we derive the conservation law

$$\partial_t u(x,t) + \partial_x \left(u(x,t)^4\right) = 0.$$

The solution is constant along the lines in directions $(4u^3, 1)$. The characteristics have the form

$$\partial_t x(t) = 4u(x,t)^3.$$

This has the general form of a conservation law with flux $f(u) = u^4$. We get Burgers' equation if we replace the exponent by 2, and we get a transport equation if we replace the exponent by 1.

Exercise 3
An example for a nonlinear wave equation is

$$\partial_{tt} E = \partial_x \left(|\partial_x E|^{p-2} \partial_x E\right)$$

where $p \geq 1$. Reformulate this as a system of conservation laws.

Solution 3
Similar as in the lecture, we introduce the variables $v = \partial_t E$ and $s = \partial_s E$, which represent the velocity and the slope. The system of conservation laws consists of

$$\partial_t s = \partial_x v, \qquad \partial_t v = \partial_x \left(s^{p-2} s\right).$$

In matrix form, we therefore find

$$\partial_t \begin{pmatrix} s \\ v \end{pmatrix} = \partial_x \left(\begin{pmatrix} 0 & v \\ s^{p-2} & 0 \end{pmatrix} \begin{pmatrix} s \\ v \end{pmatrix} \right).$$

Exercise 4
Consider the transport equation

$$\partial_t u(x,t) + \partial_x \left(a(t) u(x,t)\right) = 0$$

where the velocity depends only one the time variable t. Find the characteristics and show that the characteristics do not intersect.

Suppose that u describes the density of cars along a one-dimensional street. What sort of traffic flow does this conservation law describe? *This simple PDE is only for practice purposes.*

Solution 4
The characteristic originating at x_0 satisfies

$$\partial_t x(t) = a(t), \quad x(0) = x_0.$$

Consequently, it is given by

$$x(t) = x_0 + \int_0^t a(s) ds,$$

Suppose that $x_0(t)$ and $x_1(t)$ are two characteristics that start at x_0 and x_1, respectively, and suppose that $x_1 > x_0$.

We then have

$$x_1(t) - x_0(t) = x_1 + \int_0^t a(s) ds - x_0 + \int_0^t a(s) ds = x_1 - x_0.$$

So the distance between $x_1(t)$ and $x_0(t)$ remains constant over time.

If we interpret this as a model for traffic flow, then it describes a flow of cars where everyone drives exactly the same speed at every time, but the speed may change over time (and everyone will obey the speed rule). Obviously, this is quite only a simplified traffic flow model.

Exercise 5
(Hard) Consider the conservation law (transport equation)

$$\partial_t u(x,t) + \partial_x \left(a(x) u(x,t)\right) = 0$$

We have seen that the characteristics satisfy the following ordinary differential equation with initial values:

$$\partial_t x(t) = a(x(t)), \quad x(0) = x_0.$$

Show that the characteristics do not intersect if the velocity is always positive. [1]

[1] This is often noted casually in the literature, but can be proven rigorously.

Solution 5

Suppose that x_0 and x_1 are two points with $x_0 < x_1$. We write $x_0(t)$ and $x_1(t)$ for the two associated characteristics. They satisfy the equations

$$x_0(t) = x_0 + \int_0^t x_0'(s)ds = x_0 + \int_0^t a(x_0(s))ds, \qquad x_1(t) = x_1 + \int_0^t x_1'(s)ds = x_1 + \int_0^t a(x_1(s))ds.$$

If the two characteristics intersect, then there is some T such that $x_0(T) = x_1(T)$. At the intersection time T we then must have

$$x_0 + \int_0^T a(x_0(s))ds = x_0(T) = x_1(T) = x_1 + \int_0^T a(x_1(s))ds.$$

Since the velocity is always positive, $x_0(t)$ must have have passed x_1 at some time $t_0 > 0$, where we have $x_0(t_0) = x_1$. We split the first integral

$$x_0 + \underbrace{\int_0^{t_0} a(x_0(s))ds}_{x_1 - x_0} + \underbrace{\int_{t_0}^T a(x_0(s))ds}_{\int_0^{T-t_0} a(x_1(s))ds} = x_1 + \int_0^T a(x_1(s))ds.$$

Here, the first underbraced integral uses $x_0(0) = x_0$ and $x_0(t_0) = x_1$. The second underbraced integral uses that the characteristic of $x_0(t)$ follows the same trajectory of $x_1(t)$ (but with a time delay). So this becomes

$$x_1 + \int_0^{T-t_0} a(x_1(s))ds = x_1 + \int_0^T a(x_1(s))ds.$$

But this cannot be, because a is always positive.