Exercise Set 8: Gudonov's Method

Exercise 1
Discuss qualitatively the derivation of *Godunov's method*; sketch each step in the solution process. Which part of the algorithm can make its implementation particularly difficult?

Solution 1
Godunov's method can be outlined in two steps. Suppose we have an approximation v^n of the solution u at t_n. (i) Define $\widetilde{u}^n(x,t)$ for all x and $t_n < t < t_{n+1} = t_n + k$ as the exact solution to the conservation law, satisfying the initial condition

$$\widetilde{u}^n\left(x, t_n\right) = v_j^n \quad x \in (x_{j-1/2}, x_{j+1/2}) \quad \forall j \ . \tag{1}$$

(ii) Average the resulting function $\widetilde{u}^n\left(x, t_{n+1}\right)$ over each cell $(x_{j-1/2}, x_{j+1/2})$ to obtain the approximation

$$v_j^{n+1} = \frac{1}{h}\int_{x_{j-1/2}}^{x_{j+1/2}} \tilde{u}^n\left(x, t_{n+1}\right) \mathrm{d}\, x \tag{2}$$

at t_{n+1}. Now this procedure can be repeated to advance to the next time-step.

In Step (i), we need to solve an exact Riemann problem at each cell-interface, over a small time interval (t_n, t_{n+1}). Since $\widetilde{u}^n$ is a solution to the conservation law, (2) yields

$$\begin{aligned} v_j^{n+1} &= \frac{1}{h}\int_{x_{j-1/2}}^{x_{j+1/2}} \tilde{u}^n\left(x, t_n\right) \mathrm{d}\, x \qquad (3)\\ &- \frac{1}{h}\left(\int_{t_n}^{t_n+k} f\left(\tilde{u}^n\left(x_{j-1/2}, t\right)\right) \mathrm{d}\, t - \int_{t_n}^{t_n+k} f\left(\tilde{u}^n\left(x_{j-1/2}, t\right)\right) \mathrm{d}\, t\right) \ . \qquad (4) \end{aligned}$$

First notice that

$$v_j^n = \frac{1}{h}\int_{x_{j-1/2}}^{x_{j+1/2}} \tilde{u}^n\left(x, t_n\right) \mathrm{d}\, x \ , \tag{5}$$

since $\widetilde{u}^n$ satisfies (1). Now, let us look at the other two integrals in (3). Because $\widetilde{u}^n\left(\cdot, t_n\right)$ is piecewise constant, with a discontinuity at each $x_{j-1/2}$, and given that the time step k is sufficiently small, $\widetilde{u}^n\left(x_{j-1/2}, \cdot\right)$ is constant. Let

$$\tilde{u}^n\left(x_{j-1/2}, t\right) = u^*\left(v_{j-1}^n, v_j^n\right) \quad t \in (t_n, t_{n+1}) \ . \tag{6}$$

So, if we take

$$F(u_l, u_r) = f\left(u^*(u_l, u_r)\right) = \frac{1}{k}\int_{t_n}^{t_n+k} f\left(u^*(u_l, u_r)\right) \mathrm{d}\, t \ , \tag{7}$$

then (2) becomes

$$v_j^{n+1} = v_j^n - \frac{k}{h}\Big[F\left(v_{j+1}^n, v_j^n\right) - F\left(v_j^n, v_{j-1}^n\right)\Big] \ . \tag{8}$$

This is how the method is applied in practice, provided we can find u^*. Unfortunately, evaluating the intermediate can be very expensive, and at times impossible. This motivates the need to construct *approximate Riemann solvers*.

Exercise 2
Consider the scalar conservation law

$$u_t + f(u)_x = 0 \ , \tag{9}$$

and initial condition

$$u(x,0) = \begin{cases} u_l & x < 0 \\ u_r & 0 < x \end{cases} \ , \tag{10}$$

where the flux f is convex ($f'' > 0$). Godunov's method relies on finding the intermediate state $u^* = u^*(u_l, u_r)$ for which $u(0,t) = u^*$, for $t > 0$.

1. Show that this intermediate state is given by the following:

   1. $f'(u_l), f'(u_r) \geq 0 \implies u^* = u_l$
   2. $f'(u_l), f'(u_r) \leq 0 \implies u^* = u_r$
   3. $f'(u_l) \geq 0 \geq f'(u_r) \implies u^* = \begin{cases} u_l & s > 0 \\ u_r & s < 0 \end{cases} \ , \qquad s = \dfrac{f(u_r) - f(u_r)}{u_r - u_l}$
   4. $f'(u_l) < 0 < f'(u_r) \implies u^* = u_m \ ,$ where u_m is the solution to $f'(u_m) = 0$.

2. Use (a) to show that Godunov's flux is given by

$$F(u_l, u_r) = \begin{cases} \min\limits_{u_l \leq u \leq u_r} f(u) & u_l \leq u_r \\ \max\limits_{u_r \leq u \leq u_l} f(u) & u_l > u_r \end{cases} \ . \tag{11}$$

3. Show that Godunov's flux (15) is monotone.

Solution 2
Consider the scalar conservation law

$$u_t + f(u)_x = 0 \ , \tag{12}$$

and initial condition

$$u(x,0) = \begin{cases} u_l & x < 0 \\ u_r & 0 < x \end{cases} \ , \tag{13}$$

where the flux f is convex ($f'' > 0$). Godunov's method relies on finding the intermediate state $u^* = u^*(u_l, u_r)$ for which $u(0,t) = u^*$, for $t > 0$.

1. We show that u^* is given by

   1. $f'(u_l), f'(u_r) \geq 0 \implies u^* = u_l$
   2. $f'(u_l), f'(u_r) \leq 0 \implies u^* = u_r$
   3. $f'(u_l) \geq 0 \geq f'(u_r) \implies u^* = \begin{cases} u_l & s > 0 \\ u_r & s < 0 \end{cases} \ , \qquad s = \dfrac{f(u_r) - f(u_r)}{u_r - u_l}$
   4. $f'(u_l) < 0 < f'(u_r) \implies u^* = u_m \ ,$ where u_m is the solution to $f'(u_m) = 0$.

   Note that since f is strictly convex, the Jacobian f' is a strictly increasing function.

   Suppose $f'(u_l), f'(u_r) \geq 0$. If $u_l > u_r$, the entropy solution is a shock moving at speed given by the RH condition

$$s = \frac{f(u_l) - f(u_r)}{u_l - u_r} \tag{14}$$

and $f'(u_l) > s > f'(u_r)$ according to the entropy condition. This implies that the shock speed is positive, and thus we have $u^* = u_l$. If $u_l \le u_r$, the entropy solution is a rarefaction wave. Since $f'(u_l) > 0$, the left front of the wave moves to the right, and thus $u^* = u_l$.

Next suppose $f'(u_l), f'(u_r) \le 0$. By similar arguments we get $u^* = u_r$.

If $f'(u_l) \ge 0 \ge f'(u_r)$, the entropy solution is a shock, and the intermediate state u^* is determined by the sign of the shock speed (14).

Finally suppose $f'(u_l) < 0 < f'(u_r)$. Then, the entropy solution is a rarefaction wave. This time $x = 0$ falls inside the rarefaction fan. As we have seen in a previous exercise (see exercise set 3), the rarefaction solution is given by $u(x,t) = w\left(x/t\right)$, where w is the solution to $f'\left(w\left(\xi\right)\right) = \xi$. Since we are interested in the value of u at $x = 0$, we have $u^* = w\left(0\right)$.

2. Before we begin, let as remark that since f is strictly convex, the its maximum on a given closed interval $[u_1, u_2]$ is achieved its maximum at an end point of the interval. Furthermore, there exists a unique θ where f' vanishes. This point corresponds to the global minima of f. If $\theta \in (u_1, u_2)$, then f achieves its minimum at θ, else at an end point of the interval. To show that Godunove's flux is given by

$$F(u_l, u_r) = \begin{cases} \min\limits_{u_l \le u \le u_r} f(u) & u_l \le u_r \\ \max\limits_{u_r \le u \le u_l} f(u) & u_l > u_r \end{cases} . \tag{15}$$

we look at the different possible cases.

If $u_l > u_r$, the entropy solution is a shock. In this case, u^* is determined by the sign of s. If $f(u_l) > f(u_r)$, then $u^* = u_l$, and if $f(u_l) < f(u_r)$, then $u^* = u_r$. Either way, $f\left(u^*\right)$ is the maximum of f in $[u_r, u_l]$.

If $u_l \le u_r$, the entropy solution is a rarefaction wave. This is possible only when 1,2 or 4 are valid. If 1 is valid, f is increasing in $[u_l, u_r]$, and $f(u_l)$ is the minimum of f in this interval. Similarly, if 2 is valid, f is decreasing in $[u_l, u_r]$, and $f(u_r)$ is the minimum of f in this interval. If 4 is valid, f achieves its minimum at the internal point u_m, where $f'\left(u_m\right) = 0$.

3. To show that Godunov's flux, given by (15), is monotone, we show that it is non-decreasing in its first argument and non-increasing in its second argument. If $u_l < u_r$ and $\epsilon > 0$ is small enough, then

$$F\left(u_l + \epsilon, u_r\right) = \min_{u_l + \epsilon \le u \le u_r} f(u) \ge \min_{u_l \le u \le u_r} f(u) = F(u_l, u_r) \ , \tag{16}$$

and if $u_l \ge u_r$, then

$$F\left(u_l + \epsilon, u_r\right) = \max_{u_r \le u \le u_l + \epsilon} f(u) \ge \max_{u_r \le u \le u_l} f(u) = F(u_l, u_r) \ . \tag{17}$$

Similarly, one can show that F is non-increasing in its second argument.

Exercise 3

The purpose of this exercise is to illustrate the *Lax-Wendroff Theorem.* This theorem states that if there exists a sequence $\{(h_l, k_l)\}_l^\infty$ with $k_l = \lambda h_l$ (λ is kept constant), such that the corresponding numerical solutions $\{v_l\}_{l=1}^\infty$ obtained by a conservative method converges to some function u, then the limit u is a weak solution of the conservation law. Notice that to deduce the conclusion, we assume that v_l converges as $l \to \infty$. That is, convergence is not a conclusion of the Lax-Wendroff theorem. Also recall that in general weak solutions are not unique, so the theorem does not guarantee the limit is the correct entropy solution.

Consider a conservative method

$$v_j^{n+1} = v_j^n - \frac{k}{h}\left(F\left(v_j^n, v_{j+1}^n\right) - F\left(v_{j-1}^n, v_j^n\right)\right) \tag{18}$$

where the numerical flux F is given by

$$F\left(v, w\right) = \begin{cases} f(v) & \frac{f(v) - f(w)}{v - w} \ge 0 \\ f\left(w\right) & \frac{f(v) - f(w)}{v - w} < 0 \end{cases} . \tag{19}$$

1. Construct the entropy solution to the following initial value problem

$$u_t + \left(\frac{1}{2}u^2\right)_x = 0 \quad u(x,0) = \begin{cases} -1 & x < 1 \\ 1 & x > 1 \end{cases}. \tag{20}$$

2. Fix $k/h = 0.5$, and implement the above method to (20), in $x \in (0,2)$, $0 < t \leq 0.25$, with the initial data discretized using cell averages. On the boundaries, set $u(0,t) = -1$, and $u(2,t) = 1$.

3. Run the computations by choosing i) $h_l = \frac{2}{l}$, ii)$h_l = \frac{2}{2l}$, iii) $h_l = \frac{2}{2l+1}$, for $l \in \mathbb{N}$.

4. What can you deduce from your results regarding the each of the three sequences of numerical solutions obtained. Explain your results and conclude how they fit with the Lax-Wendroff theorem.

Solution 3 1. Since Burgers equation is convex and

$$u(x,0) = \begin{cases} -1 & x < 1 \\ 1 & x > 1 \end{cases}, \tag{21}$$

that is $u_l < u_r$, a shock is not admissible. Therefore, the entropy solution must be a rarefaction wave. The exact solution to the problem with initial data (21) is given by

$$u(x,t) = \begin{cases} -1 & x \leq -t+1 \\ \frac{x-1}{t} & -t+1 < x \leq t+1 \\ 1 & x > t+1 \end{cases}. \tag{22}$$

(b),(c) See Matlab code attached at the end of this manual. Experiment with different values of l.

2. We notice that in case of $k_l = \frac{1}{2l}$, for any choice of $l \in \mathbb{N}$, we get an approximation to the entropy solution, and that the sequence of solutions converges as $l \to \infty$. In the case $k_l = \frac{1}{2l+1}$, for any choice of $l \in \mathbb{N}$ we get the entropy violating solution of a stationary discontinuity. Therefore, the sequence of solutions obtained by taking $k_l = \frac{1}{2l+1}$ converges to the same entropy violating solution as $l \to \infty$. Note that in either case, the converged solution is still a weak solution, which is in accordance with the Lax-Wendroff theorem. However, by taking $k_l = \frac{1}{l}$, the numerical solution does not converge to any solution with the current scheme.