Exercise Set 6: More on the Finite Difference Methods

Exercise 1 (Non-conservative scheme)
Consider the non-conservative scheme for Burgers' equation:

$$U_j^{n+1} = U_j^n - \frac{k}{h} U_j^n \left(U_j^n - U_{j-1}^n \right). \tag{1}$$

Show that this scheme maps non-negative monotonically increasing sequences onto non-negative monotonically increasing sequences. *Remark: while this scheme is not conservative, is still of interest in a few contexts. The non-negativity condition is due to the upwinding implicit in the scheme.*

Solution 1
The scheme can be expressed in the following form: $U_j^{n+1} = G(U_{j-1}^n, U_j^n)$ with

$$G(V,W) = W - \frac{k}{h} W\left(W - V\right) = W\left(1 - \frac{k}{h}W\right) + \frac{k}{h} VW \tag{2}$$

This scheme maps non-negative monotonically increasing sequences onto non-negative monotonically increasing sequences provided that the sequence are non-negative and that $W \leq \frac{h}{k}$.

Exercise 2 (Monotonicity of FTCS)
What is the flux of the FTCS scheme for the transport equation? Relate it to other fluxes that you know. Determine whether the FTCS scheme is monotone.

Solution 2
The transport equation of speed $a > 0$ reads $u_t(x,t) + a u_x(x,t) = 0$ in $(x,t) \in \mathbb{R} \times [0,\infty)$ equipped with initial condition $u(x,t) = u_0(x)$, $x \in \mathbb{R}$. The FTCS (Forward in Time, Central in Space) uses the following approximations

$$\partial_t u(x_j, t^n) \approx \frac{U_j^{n+1} - U_j^n}{k} \quad \text{and} \quad \partial_x u(x_j, t^n) \approx \frac{U_{j+1}^n - U_{j-1}^n}{2h}. \tag{3}$$

Thus, the scheme FTCS reads

$$U_j^{n+1} = U_j^n - \frac{ak}{2h}\left(U_{j+1}^n - U_{j-1}^n\right) = U_j^n - \frac{k}{h}\left(a\frac{U_j^n + U_{j+1}^n}{2} - a\frac{U_{j-1}^n + U_j^n}{2}\right). \tag{4}$$

The flux in this case is $F(U,V) = a\frac{U+V}{2}$. The FTCS scheme can be expressed in the following form: $U_j^{n+1} = G(U_{j-1}^n, U_j^n, U_{j+1}^n)$ with

$$G(U,V,W) = V - \frac{ak}{2h}\left(W - U\right). \tag{5}$$

One can readily conclude that the scheme is not monotone.

Exercise 3 (Order of Convergence)
Show that the FTFS scheme for the transport equation is of order $(1,1)$ and that the FTCS scheme is of order $(1,2)$.

Solution 3
The FTFS (Forward in Time, Forward in Space) uses the following approximations

$$\partial_t u(x_j, t^n) \approx \frac{U_j^{n+1} - U_j^n}{k} + \mathcal{O}(k) \quad \text{and} \quad \partial_x u(x_j, t^n) \approx \frac{U_{j+1}^n - U_j^n}{h} + \mathcal{O}(h). \tag{6}$$

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$$\partial_t u(x_j, t^n) \approx \frac{U_j^{n+1} - U_j^n}{k} + \mathcal{O}(k) \quad \text{and} \quad \partial_x u(x_j, t^n) \approx \frac{U_{j+1}^n - U_{j-1}^n}{2h} + \mathcal{O}(h^2). \tag{7}$$

For a complete argument, we refer to the Slides 9 "Error analysis of finite-difference schemes".

Exercise 4 (Linear Scheme)
Under what conditions is a linear scheme $F(U,V) = w_1 U + w_2 V$ a monotone scheme? Describe the discrete Kruzkov entropy entropy-flux pairs.

Solution 4
To show that the flux $F(U,V) = w_1 U + w_2 V$ leads to a monotone scheme, we need to show that

$$\lambda \partial_1 F(U_{j-1}^n, U_j^n) \geq 0 \tag{8}$$

$$-\lambda \partial_2 F(U_j^n, U_{j+1}^n) \geq 0 \tag{9}$$

$$1 - \lambda(\partial_1 F(U_j^n, U_{j+1}^n) - \partial_2 F(U_{j-1}^n, U_j^n)) \geq 0 \tag{10}$$

where $\lambda = k/h$. The first condition implies that $w_1 \geq 0$, and the second that $w_2 \leq 0$, whereas the third that

$$1 - \lambda\,(w_1 - w_2) \geq 0 \tag{11}$$

Exercise 5 (Roe flux)
Show that the Roe flux can be written as

$$F(U,V) = \frac{f(V) + f(U)}{2} + \frac{1}{2}\,\mathrm{sgn}(V - U)|f(V) - f(U)|. \tag{12}$$

Either show that the Roe flux is monotonicity-preserving or give a counter example.

Solution 5
The Roe flux is defined as

$$F(U,V) = \begin{cases} f(U), & s \geq 0 \\ f(V), & s < 0 \end{cases}, \quad s = \frac{f(U) - f(V)}{U - V}. \tag{13}$$

For Burger's equation, one has $f(U) = \frac{U^2}{2}$ and

$$\frac{\partial F}{\partial U}(U,V) = \begin{cases} U, & U + V \geq 0, \\ 0, & U + V < 0, \end{cases} \tag{14}$$

By selecting $U < 0 < V$ monotonicity cannot be guaranteed.