

Exercise Set 3: Characteristics, Entropy and Weak solutions

Exercise 1 (Entropy Solutions)

Consider the conservation law

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0, \quad u(x, 0) = u_0(x), \quad (1)$$

and $f(u) = u^2/2$. Consider the following initial conditions

$$(a) \quad u_0(x) = \begin{cases} 1, & \text{for } x < -1, \\ 0, & \text{for } -1 < x < 1, \\ -1, & \text{for } x > 1 \end{cases} \quad \text{and} \quad (b) \quad u_0(x) = \begin{cases} -1, & \text{for } x < -1, \\ 0, & \text{for } -1 < x < 1, \\ 1, & \text{for } x > 1. \end{cases} \quad (2)$$

Draw the profile of $u_0(x)$ and sketch the characteristics of the entropy solution $u(x, t)$ in the $x - t$ plane, and determine $u(x, t)$ for all $t > 0$.

Solution 1

We start with the solution driven by the initial data

$$u_0(x) = \begin{cases} 1 & x < -1 \\ 0 & -1 < x < 1 \\ -1 & x > 1 \end{cases} . \quad (3)$$

Figure 1 contains a sketch of the initial data profile $u_0(x)$ with characteristics evolving. We see that two discontinuities are present in the initial data. The task is to determine the exact solution $u(x, t)$ for all $t > 0$. For Burger's equation, a shock forms if $u_l > u_r$. Thus, both discontinuities in the initial function produce shocks. To determine the speed of each shock we use the Rankine-Hugoniot jump condition

$$s = \frac{f(u_l) - f(u_r)}{u_l - u_r} . \quad (4)$$

Applied to each discontinuity in the initial data, we get

$$s_1 = \frac{\frac{1}{2}1^2 - \frac{1}{2}0^2}{1 - 0} = \frac{1}{2} \quad s_2 = \frac{\frac{1}{2}0^2 - \frac{1}{2}(-1)^2}{0 - (-1)} = -\frac{1}{2} . \quad (5)$$

The left shock moves to the right and the right shock moves to the left. From this we expect the two shocks to meet at $x = 0$ at time $t = 2$. We can now write the exact solution up until $t = 2$:

$$u(x, t) = \begin{cases} 1 & x < \frac{1}{2}t - 1 \\ 0 & \frac{1}{2}t - 1 < x < -\frac{1}{2}t + 1 \\ -1 & -\frac{1}{2}t + 1 < x \end{cases} . \quad (6)$$

When the two shocks meet we are left with a Riemann problem. This time, the “initial condition” is the solution we have calculated at $t = 2$. Again we turn to the Rankine-Hugoniot jump condition. This time, the shock's speed is

$$s_3 = \frac{\frac{1}{2}1^2 - \frac{1}{2}(-1)^2}{1 - (-1)} = 0 . \quad (7)$$

When the two shocks meet they stop moving for all time hereafter. Thus, for $t > 2$,

$$u(x, t) = \begin{cases} 1 & x < 0 \\ -1 & 0 < x \end{cases} . \quad (8)$$

Now we move on to the solution for the initial data set

$$u_0(x) = \begin{cases} -1 & x < -1 \\ 0 & -1 < x < 1 \\ 1 & 1 < x \end{cases} . \quad (9)$$

In Figure 2 is a sketch of the initial data profile $u_0(x)$ with characteristics evolving as rarefaction waves. A general solution model for a rarefaction wave on a Riemann problem is supplied in the exercise description for initial data centered at zero. Using the solution model with $f'(v(\xi)) = \xi \Rightarrow v(\xi) = \xi$ we arrive at

$$u(x, t) = \begin{cases} -1 & x < -t - 1 \\ \frac{x+1}{t} & -t - 1 < x < -1 \\ 0 & -1 < x < 1 \\ \frac{x-1}{t} & 1 < x < t + 1 \\ 1 & t + 1 < x \end{cases} . \quad (10)$$

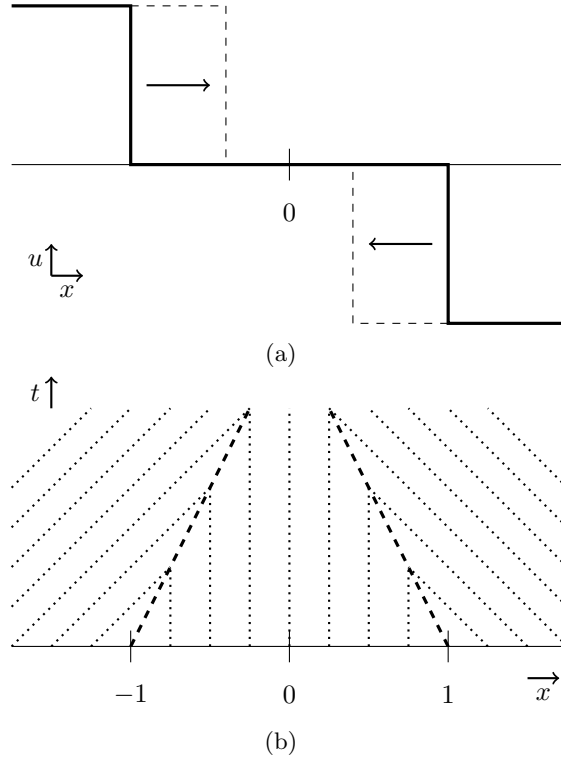


Figure 1: Initial data u_0 given by (3), applied to the inviscid burgers equation. The two shocks move towards each other and merge at $x = 0$. At $x = 0$ they form a new stationary shock. (a) Initial data. (b) Characteristics.

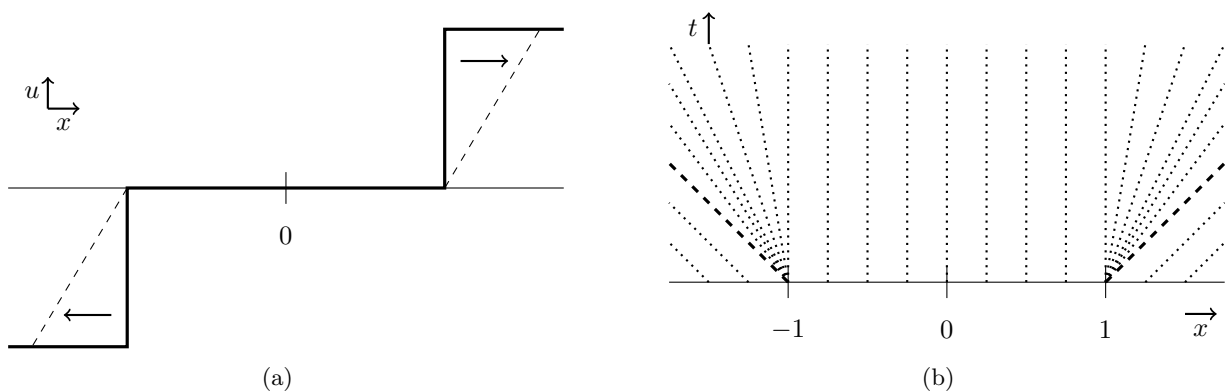


Figure 2: Exercise 3.1b. Initial data u_0 given by (9), applied to the inviscid burgers equation. The two rarefaction waves move away from each other with equal speed and never merge. (a) Initial data. (b) Characteristics.

Exercise 2 (Traffic Flow Equation)

Consider the traffic flow equation

$$\frac{\partial q}{\partial t} + \frac{\partial}{\partial x} f(q) = 0, \quad (11)$$

where $q(x, t)$ is the car density, $U(q)$ is the traffic speed as a function of the car density and $f(q) = qU(q)$. A simple model for traffic flow is obtained by taking

$$U(q) = u_m \left(1 - \frac{q}{q_m} \right), \quad (12)$$

with $u_m > 0$ and q_m being the maximum speed and maximum car density, respectively. If the car density is maximal, we say that the traffic is “bumper-to-bumper”. At zero density (empty road), the traffic speed is u_m . As q approaches q_m , the speed decreases to zero. The model then reads

$$\frac{\partial q}{\partial t} + \frac{\partial}{\partial x} u_m \left(q - \frac{q^2}{q_m} \right) = 0. \quad (13)$$

We equip (13) with the initial condition

$$q(x, 0) = \begin{cases} q_l, & \text{for } x < 0, \\ q_r, & \text{for } x > 0. \end{cases} \quad (14)$$

- (a) **The green light problem.** Assume that the traffic is standing at a red light, placed at $x = 0$, while the road ahead is empty. At time $t = 0$, the traffic light turns green and we want to describe the car flow evolution for $t > 0$. To represent this situation, we set $q_l = q_m$ and $q_r = 0$ in (14). Draw the profile of $q_0(x)$ and sketch the characteristics of the entropy solution $q(x, t)$ in the $x - t$ plane, and determine $u(x, t)$ for all $t > 0$.
- (b) **Traffic jam ahead.** We now consider the initial density profile with $q_l = \frac{1}{8}q_m$ and $q_r = q_m$. For $x > 0$, the density is maximal and therefore the traffic is “bumper-to-bumper”. The cars on the left move with speed $U = \frac{7}{8}u_m$, so that we expect congestion. Draw the profile of $q_0(x)$ and sketch the characteristics of the entropy solution $q(x, t)$ in the $x - t$ plane, and determine $q(x, t)$ for all $t > 0$.
- (c) **Entropy Solutions.** Show that for the traffic flow equation (13), the condition $q_l < q_r$ is required for a shock to be admissible. Do this by verifying each of the following conditions:
 - (i) The entropy condition $f'(q_l) > s > f'(q_r)$.
 - (ii) There exists an entropy function $\eta(q)$ and a corresponding entropy flux $\psi(q)$ such that

$$s(\eta(q_r) - \eta(q_l)) \geq \psi(q_r) - \psi(q_l) \quad (15)$$

holds if and only if $q_l < q_r$.

Solution 2

The green light problem. Let us set $F(x, t, z) = q(x, t) - z$. Observe that (13) can be written as

$$\frac{\partial q}{\partial t} + u_m \left(1 - 2 \frac{q}{q_m} \right) \frac{\partial q}{\partial x} = 0. \quad (16)$$

Hence, it follows that

$$\begin{pmatrix} \frac{\partial q}{\partial t} \\ \frac{\partial q}{\partial x} \\ -1 \end{pmatrix} \begin{pmatrix} 1 \\ u_m \left(1 - 2 \frac{q}{q_m} \right) \\ 0 \end{pmatrix} \quad (17)$$

The vector $(\frac{\partial q}{\partial t}, \frac{\partial q}{\partial x} - 1)^\top$ is normal to the surface defined implicitly by $F(t, x, z) = 0$. Starting from a given point $(0, \xi, q_0(\xi))$ we consider the characteristic curve with speed vector $(1, u_m \left(1 - 2 \frac{q}{q_m} \right), -1)^\top$. Consider the parametrization $(t(s), x(s), z(s))^\top$ of the characteristic curve. Now, we enforce its tangent vector to match $(\frac{\partial q}{\partial t}, \frac{\partial q}{\partial x} - 1)^\top$, thus leading to the following set of ODEs:

$$\frac{dt(s)}{ds} = 1, \quad \frac{dx(s)}{ds} = u_m \left(1 - 2 \frac{q}{q_m} \right), \quad \frac{dz(s)}{ds} = 0 \quad (18)$$

Considering the starting point $(0, \xi, q_0(\xi))$ of the characteristic curve, we get

$$t(s) = s, \quad z(s) = q_0(\xi) \quad (19)$$

Hence, the solution to the problem is constant along the characteristic curve. This implies that.

$$x(t) = -u_m t + \xi, \text{ for } \xi \leq 0, \quad x(t) = u_m t + \xi, \text{ for } \xi > 0. \quad (20)$$

In addition, in this case we observe the formation of a rarefaction wave in between the two constant states, whose expression is

$$q(x, t) = \begin{cases} q_m & x < -u_m t \\ \frac{q_m}{2} \left(1 - \frac{x}{u_m t} \right) & -u_m t < x < u_m t \\ 0 & x \geq u_m t \end{cases} \quad (21)$$

Traffic jam ahead. In this case, the characteristics are given by

$$x = \frac{3}{4} u_m t + \xi, \text{ for } \xi \leq 0, \quad x = -u_m t + \xi, \text{ for } \xi < 0, \quad (22)$$

In this case the characteristics intersect somewhere in finite time and the solution $q(x, t)$ becomes “multivalued”. In other words, the solution to the problem admits two different values at the same point. In this scenario, the solution has a discontinuity and a shock propagates along this discontinuity. The speed at which this shock wave propagates is governed by the Rankine-Hugoniot condition:

$$\dot{s}(t) = \frac{f(q_r) - f(q_l)}{q_r - q_l} = -\frac{1}{8} u_m. \quad (23)$$

The solution is

$$q(x, t) = \begin{cases} \frac{q_m}{8}, & x < -\frac{1}{8} u_m t \\ q_m & x > -\frac{1}{8} u_m t. \end{cases}$$

Entropy Solutions. For the sake of simplicity, we assume $q_m = 1$. The flux function f is given by $f(q) = u_m (q - q^2)$. We choose $\eta(q) = q^2$ as an entropy function. By

$$\psi'(q) = \eta'(q) f'(q)$$

we deduce the corresponding entropy flux to be

$$\psi(q) = u_m \left(q^2 - \frac{4}{3} q^3 \right).$$

Inserting ψ and η into

$$s(\eta(q_r) - \eta(q_l)) > \psi(q_r) - \psi(q_l) \quad (24)$$

we find that

$$u_m(1 - (q_r + q_l))(q_r^2 - q_l^2) > u_m\left(q_r^2 - \frac{4}{3}q_r^3\right) - u_m\left(q_l^2 - \frac{4}{3}q_l^3\right) \quad (25)$$

which after some algebra can be shown to be equivalent to

$$0 > \frac{1}{3}(q_l - q_r)^3. \quad (26)$$

Thus, we reach the same conclusion: a shock is admissible if $q_l < q_r$.

Exercise 3 (Liu's Entropy Condition)

Consider the conservation law

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0 \quad (27)$$

where $f(u) = e^u$. We discuss the Riemann problem

$$u_0(x) = \begin{cases} u_l, & \text{for } x < 0, \\ u_r, & \text{for } x > 0 \end{cases} \quad (28)$$

1. Show that when $u(x, t)$ is a weak solution and $\lambda > 0$, then $u(\lambda x, \lambda t)$ is a weak solution too.
2. Suppose that the solution has a shock wave with values u_l and u_r to the left and to the right, respectively. Which values u_l and u_r are admissible on the basis of Liu's entropy condition?

Remark: the scale invariance above suggests that "physical" solutions to the Riemann problem should be scale-invariant themselves.

Solution 3

If $u(x, t)$ is a weak solution then for any compactly supported smooth function φ in $\mathbb{R} \times [0, \infty)$ it holds

$$\int_0^\infty \int_{\mathbb{R}} [u\varphi_t + f(u)\varphi_x] dx dt + \int_{\mathbb{R}} u_0(x)\varphi(x, 0) dx = 0. \quad (29)$$

Consider the change of variables $x = \lambda\tilde{x}$ and $t = \lambda\tilde{t}$, then $dx = \lambda d\tilde{x}$ and $t = \lambda d\tilde{t}$ and

$$\frac{\partial \varphi}{\partial t}(\lambda\tilde{x}, \lambda\tilde{t}) = \frac{1}{\lambda} \frac{\partial \tilde{\varphi}}{\partial \tilde{t}}(\tilde{x}, \tilde{t}) \quad \text{and} \quad \frac{\partial \varphi}{\partial x}(\lambda\tilde{x}, \lambda\tilde{t}) = \frac{1}{\lambda} \frac{\partial \tilde{\varphi}}{\partial \tilde{x}}(\tilde{x}, \tilde{t}), \quad (30)$$

where $\tilde{\varphi}(\tilde{x}, \tilde{t}) = \varphi(\lambda\tilde{x}, \lambda\tilde{t})$. Then

$$\int_0^\infty \int_{\mathbb{R}} [u(x, t)\varphi_t(x, t) + f(u(x, t))\varphi_x(x, t)] dx dt = \lambda^2 \int_0^\infty \int_{\mathbb{R}} [u(\lambda\tilde{x}, \lambda\tilde{t})\varphi_t(\lambda\tilde{x}, \lambda\tilde{t}) + f(u(\lambda\tilde{x}, \lambda\tilde{t}))\varphi_x(\lambda\tilde{x}, \lambda\tilde{t})] d\tilde{x} d\tilde{t} \quad (31)$$

$$= \lambda \int_0^\infty \int_{\mathbb{R}} \left[u(\lambda\tilde{x}, \lambda\tilde{t}) \frac{\partial \tilde{\varphi}}{\partial \tilde{t}}(\tilde{x}, \tilde{t}) + f(u(\lambda\tilde{x}, \lambda\tilde{t})) \frac{\partial \tilde{\varphi}}{\partial \tilde{x}}(\tilde{x}, \tilde{t}) \right] d\tilde{x} d\tilde{t} \quad (32)$$

and

$$\int_{\mathbb{R}} u_0(x)\varphi(x, 0) dx = \lambda \int_{\mathbb{R}} u(\lambda\tilde{x}, \lambda\tilde{t})\tilde{\varphi}(\tilde{x}, 0) d\tilde{x}. \quad (33)$$

Then, since $\lambda > 0$ and $u(x, t)$ is a weak solution

$$\lambda \int_0^\infty \int_{\mathbb{R}} \left[u(\lambda\tilde{x}, \lambda\tilde{t}) \frac{\partial \tilde{\varphi}}{\partial \tilde{t}}(\tilde{x}, \tilde{t}) + f(u(\lambda\tilde{x}, \lambda\tilde{t})) \frac{\partial \tilde{\varphi}}{\partial \tilde{x}}(\tilde{x}, \tilde{t}) \right] d\tilde{x} d\tilde{t} + \lambda \int_{\mathbb{R}} u(\lambda\tilde{x}, \lambda\tilde{t})\tilde{\varphi}(\tilde{x}, 0) d\tilde{x} = 0, \quad (34)$$

thus rendering $u(\lambda x, \lambda t)$ a weak solution as well.

Liu's entropy condition means that a shock is admissible if the states u_l and u_r satisfy the condition

$$\frac{f(u^*) - f(u_l)}{u^* - u_l} \geq \frac{f(u_r) - f(u^*)}{u_r - u^*} \quad (35)$$

where $u^* \in (\min\{u_l, u_r\}, \max\{u_l, u_r\})$. This condition reflects that the characteristics must run into the shock. Requiring that the jump behind travels faster than the jump ahead, and considering that the flux $f(u) = e^u$ is strictly convex, we have that to have an admissible shock one requires $u_l < u_r$.