Exercise Set 2: More on Characteristics and Weak solutions

Exercise 1 (Method of Characteristics I)
Suppose that the flux $f(u, x, t)$ is differentiable in all variables. Find curves along which the conservation law

$$\frac{\partial u(x,t)}{\partial t} - x\frac{\partial f(u(x,t),x,t)}{\partial x} = 0 \tag{1}$$

can be written as a collection of ordinary differential equations.

Solution 1
Let us calculate

$$\frac{\partial f(u,x,t)}{\partial x} = \frac{\partial f(u,x,t)}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f(u,x,t)}{\partial x}. \tag{2}$$

Therefore,

$$\frac{\partial u(x,t)}{\partial t} - x\left(\frac{\partial f(u,x,t)}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f(u,x,t)}{\partial x}\right) = 0 \tag{3}$$

Firstly, we set $F(t, x, z) = u(x, t) - z$, hence the solution of the conservation law

$$\frac{\partial u}{\partial t} - x\frac{\partial f(u,x,t)}{\partial u}\frac{\partial u}{\partial x} = x\frac{\partial f(u,x,t)}{\partial x} \tag{4}$$

can be understood as the surface implicitly defined as $F(t, x, z) = 0$. Observe that 4 can be written as

$$\begin{pmatrix} \frac{\partial u}{\partial t} \\ \frac{\partial u}{\partial x} \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -x\frac{\partial f(u,x,t)}{\partial u} \\ x\frac{\partial f(u,x,t)}{\partial x} \end{pmatrix} = 0. \tag{5}$$

The vector $(\frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}, -1)^\top$ is normal to the surface $F(t, x, z) = 0$, hence $(1, -x\frac{\partial f(u,x,t)}{\partial u}, x\frac{\partial f(u,x,t)}{\partial x})^\top$ is a tangent vector to $F(t, x, z) = 0$. Starting from the point $(0, \xi, u_0(\xi)$ we consider the characteristic curve with speed vector $(1, -x\frac{\partial f(u,x,t)}{\partial u}, x\frac{\partial f(u,x,t)}{\partial x})^\top$. Considering a parametrization $(t(s), x(s), z(s))^\top$ with $s \in \mathbb{R}$ of the sought curve, we enforce its tangent vector to match $(1, -x\frac{\partial f(u,x,t)}{\partial u}, x\frac{\partial f(u,x,t)}{\partial x})^\top$. In doing so, we get the following set of ODEs describing the characteristic curve

$$\frac{dt(s)}{ds} = 1, \quad \frac{dx(s)}{ds} = -x\frac{\partial f(u,x,t)}{\partial u}, \quad \text{and} \quad \frac{dz(s)}{ds} = x\frac{\partial f(u,x,t)}{\partial x}. \tag{6}$$

Exercise 2 (Method of Characteristics II)

(i) Consider the conservation law

$$\frac{\partial u}{\partial t} - x\frac{\partial u}{\partial x} = 0 \tag{7}$$

with initial value

$$u(x, 0) = x. \tag{8}$$

Sketch the characteristics up to time $t = 1$. Describe the graph of the function $u(\cdot, t)$ as t increases.

(ii) Consider the conservation law

$$\frac{\partial u}{\partial t} + x\frac{\partial u}{\partial x} = 0 \tag{9}$$

with initial value

$$u(x,0) = x. \tag{10}$$

Draw the characteristics and describe the graph of the function $u(\cdot, t)$ as t increases.

Solution 2
We start by tacking (i). Firstly, we set $F(t,x,z) = u(x,t) - z$, hence the solution of the conservation law

$$\frac{\partial u}{\partial t} - x\frac{\partial u}{\partial x} = 0 \tag{11}$$

can be understood as the surface implicitly defined as $F(t,x,z) = 0$. Observe that 11 can be written as

$$\begin{pmatrix} \frac{\partial u}{\partial t} \\ \frac{\partial u}{\partial x} \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -x \\ 0 \end{pmatrix} = 0. \tag{12}$$

The vector $(\frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}, -1)^\top$ is normal to the surface $F(t,x,z) = 0$, hence $(1,-x,0)^\top$ is a tangent vector to $F(t,x,z) = 0$. Starting from a point $(0, \xi, u_0(\xi)$ we consider the characteristic curve with speed vector $(1,-x,0)^\top$. Considering a parametrization $(t(s), x(s), z(s))^\top$ with $s \in \mathbb{R}$ of the sought curve, we enforce its tangent vector to match $(1,-x,0)^\top$. In doing so, we get the following set of ODEs describing the characteristic curve

$$\frac{dt(s)}{ds} = 1, \quad \frac{dx(s)}{ds} = -x, \quad \text{and} \quad \frac{dz(s)}{ds} = 0. \tag{13}$$

Considering the aforementioned initial condition, we get the following solutions

$$t(s) = s, \quad x(s) = \xi \exp(-s), \quad \text{and} \quad z(s) = u_0(\xi). \tag{14}$$

We remark at this point that the parameters s it is actually equal to the temporal variable t, and that the solution $u(x,t)$ is constant along the characteristics and equal to the initial condition. From the implicit definition of the solution of the conservation law as $F(t,x,z) = 0$, we get that

$$u(x,t) = u_0(\xi) = u_0(x\exp(t)) = x\exp(t). \tag{15}$$

It is straightforward to verify that $u(x,t)$ in (15) is actually a solution of (11) in the classical sense. For any $\xi \in \mathbb{R}$, the characteristics in the $x - t$ plane are given by

$$t(x) = -\log\left(\frac{x}{\xi}\right), \quad \text{for} \quad x\xi > 0. \tag{16}$$

We proceed to tackle (ii). As in the previous case, we consider a parametrization $(t(s), x(s), z(s))^\top$ with $s \in \mathbb{R}$ of the sought curve, we enforce its tangent vector to match $(1, x, 0)^\top$. We get the following set of ODEs describing the characteristic curve

$$\frac{dt(s)}{ds} = 1, \quad \frac{dx(s)}{ds} = x, \quad \text{and} \quad \frac{dz(s)}{ds} = 0. \tag{17}$$

The solution to the ODEs are

$$t(s) = s, \quad x(s) = \xi \exp(s), \quad \text{and} \quad z(s) = u_0(\xi). \tag{18}$$

Again, the parameter s it is actually equal to the temporal variable t, and the solution $u(x,t)$ is constant along the characteristics. From the implicit definition of the solution of the conservation law as $F(t,x,z) = 0$, we get that

$$u(x,t) = u_0(\xi) = u_0(x\exp(-t)) = x\exp(-t). \tag{19}$$

It is straightforward to verify that $u(x,t)$ in (19) solves

$$\frac{\partial u}{\partial t} + x\frac{\partial u}{\partial x} = 0. \tag{20}$$

For any $\xi \in \mathbb{R}$, the characteristics in the $x - t$ plane are given by

$$t(x) = \log\left(\frac{x}{\xi}\right), \quad \text{for} \quad x\xi > 0. \tag{21}$$

Exercise 3 (Weak Solutions of the Linear Transport Equation)
Show that a weak solution to the linear transport equation

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0,$$

with $a \in \mathbb{R}$ and initial data

$$u(x,0) = \begin{cases} 1, & \text{for} \quad x < 0, \\ 0, & \text{for} \quad x > 0, \end{cases} \tag{22}$$

is given by

$$u(x,t) = \begin{cases} 1, & \text{for} \quad x < at, \\ 0, & \text{for} \quad x > at, \end{cases} \tag{23}$$

Solution 3
We need to prove that for all test functions $\phi \in C^1(\mathbb{R} \times [0,\infty))$ with compact support, it holds

$$\int_0^\infty \int_{-\infty}^\infty \left(\frac{\partial \phi}{\partial t} u + \frac{\partial \phi}{\partial x} f(u)\right) \mathrm{d}x\mathrm{d}t = -\int_{-\infty}^\infty \phi(x,0)u(x,0)\mathrm{d}x, \tag{24}$$

where $f(u) = au$ in the case of this exercise. Assume without loss of generality that $a > 0$. Then we calculate

$$\int_0^\infty \int_{-\infty}^\infty \frac{\partial \phi}{\partial t} u \,\mathrm{d}x\mathrm{d}t = \int_{-\infty}^0 \int_0^\infty \frac{\partial \phi}{\partial t}\mathrm{d}t\mathrm{d}x + \int_0^\infty \int_{x/a}^\infty \frac{\partial \phi}{\partial t}\mathrm{d}t\mathrm{d}x = -\int_{-\infty}^0 \phi(x,0)\mathrm{d}x - \int_0^\infty \phi\left(x, \frac{x}{a}\right)\mathrm{d}x, \tag{25}$$

and

$$\int_0^\infty \int_{-\infty}^\infty \frac{\partial \phi}{\partial x} f(u)\mathrm{d}x\mathrm{d}t = a\int_0^\infty \int_{-\infty}^{at} \frac{\partial \phi}{\partial x}\mathrm{d}x\mathrm{d}t = a\int_0^\infty \phi\left(at, t\right)\mathrm{d}t \tag{26}$$

Observing that

$$a\int_0^\infty \phi\left(at,t\right)\mathrm{d}t = \int_0^\infty \phi\left(x, \frac{x}{a}\right)\mathrm{d}x, \tag{27}$$

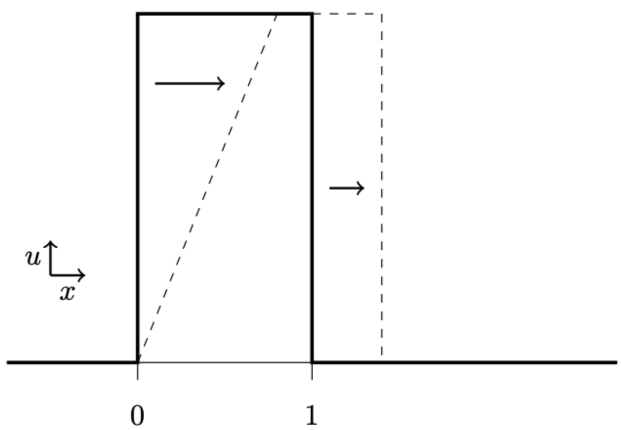
and that

$$\int_{-\infty}^0 \phi(x,0)\mathrm{d}x = \int_{-\infty}^\infty \phi(x,0)u(x,0)\mathrm{d}x, \tag{28}$$

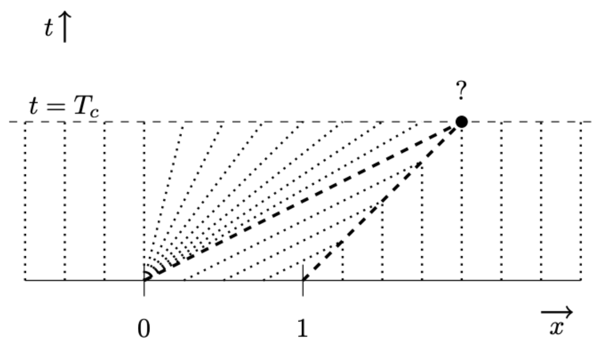
we conclude the desired result.

Exercise 4 (Rarefaction Waves)
Consider the initial value problem

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0, \quad u(x,0) = u_0(x), \tag{29}$$



(a) Initial data u_0, given by (30). The rarefaction wave and the shock both move in the positive direction.



(b) Characteristics of the solution of the Burger's equation up until time T_c. The rarefaction wave moves faster than the shock and at some point in time $t = T_c > 0$ they meet, and the characteristics cross each other.

Figure 1: Initial condition u_0 and characteristics up until T_c.

with $f(u) = \frac{u^2}{2}$, and

$$u_0(x) = \begin{cases} 2, & 0 < x < 1, \\ 0, & \text{otherwise}, \end{cases} \tag{30}$$

Here a rarefaction wave arises at one discontinuity and a shock at the other. The goal of this exercise is to determine the exact solution for all $t > 0$. In this setup, the rarefaction wave catches up with the shock at some time $T_c > 0$.

(i) Draw the profile of $u_0(x)$ and sketch the characteristics in the strip $0 < t < T_c$ of the $x - t$ plane.

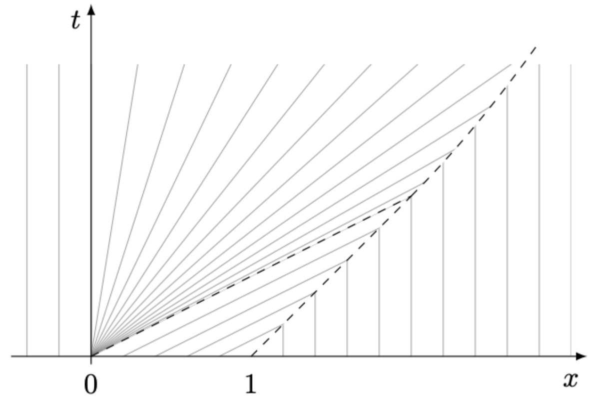
(ii) Determine the exact solution for $0 < t < T_c$.

(iii) Let $x_s(t)$ be shock's location at $t > T_c$. By using the Rankine-Hugoniot jump condition construct an ODE to determine $x_s(t)$ for all $t > T_c$. In the sketch you drew in (i), extend the characteristic lines to $t > T_c$.

Solution 4 (a) Figure 1 contains a sketch of the initial data profile u_0 with characteristics evolving. To determine the speed of the shock originating at $x = 1$, we use the Rankine-Hugoniot jump condition

$$s_{shock} = \frac{f(u_l) - f(u_r)}{u_l - u_r} = \frac{\frac{1}{2}2^2 - \frac{1}{2}(0)^2}{2 - (0)} = 1. \tag{31}$$

At the discontinuity at $x = 0$ we expect a rarefaction wave to arise. From the general solution to the rarefaction wave, we have that the right front of this wave moves with a speed of $s_{rf} = f'(u_r) = 2$. The rarefaction wave thus moves to the right twice as fast as the shock wave and at some point in time the waves must meet.



(a) Initial data u_0, given by (30). The rarefaction wave and the shock both move in the positive direction.

Figure 2: The characteristic curves (gray lines) of u in the $x-t$ plane.

(b) First we seek to determine the exact solution for $0 < t < T_c$, where T_c is the time when the rarefaction wave catches up with the shock. The time T_c is trivial to find: Since the position of the rarefaction front is $x_{rf} = 2t$ and the position of the shock is $x_{\text{shock}} = t + 1$, we have $T_c + 1 = 2T_c$, namely $T_c = 1$. Thus,

$$u(x,t) = \begin{cases} 0 & x < 0 \\ \frac{x}{t} & 0 < x < 2t \\ 2 & 2t < x < t+1 \\ 0 & t+1 < x \end{cases}$$

$$s_{shock} = \frac{f(u_l) - f(u_r)}{u_l - u_r} = \frac{\frac{1}{2}2^2 - \frac{1}{2}(0)^2}{2 - (0)} = 1$$

(c) What happens at $t > T_c$? We again use the Rankine-Hugoniot jump condition, now to construct an ODE for the position of the shock x_s after the rarefaction and shock wave have merged:

$$\frac{dx_s(t)}{dt} = \frac{f(u_l) - f(u_r)}{u_l - u_r} = \frac{\frac{1}{2}\left(\frac{x_s(t)}{t}\right)^2 - \frac{1}{2}(0)^2}{\frac{x_s(t)}{t} - (0)} = \frac{1}{2}\frac{x_s(t)}{t}$$

This ODE has the general solution $x_s(t) = C\sqrt{t}$. At $t = 1$ we know that $x_s = 2$ so $C = 2$. But what about the profile of $u(x,t)$? For times $t > T_c$, we expect the solution to the left of the shock-curve to be obtained from the rarefaction solution, i.e., $u = x/t$, while the solution on the right to be $u = 0$. Note that this is compatible with the entropy condition for a shock. Thus, the solution can be expressed as

$$u(x,t) = \begin{cases} 0 & x < 0 \\ \frac{x}{t} & 0 < x < 2\sqrt{t} \\ 0 & x > 2\sqrt{t} \end{cases}$$

Figure 2 shows the characteristic curves in the $x-t$ plane up to $t = T_c$ and beyond. One can see that the information originating from $(x,t) = (0,0)$ reaches up to the shock, even after the rarefaction front have caught up with it. This means that for $t > T_c$, in $0 < x < x_s$, the solution is determined exclusively by the rarefaction wave, and is not affected by the location, or the existence of the shock.