Exercise Set 13: Discontinuous Galerkin

Let $=_x \times_t$, where $_x = (-1, 1)$ is the spacial domain and $_t = (0, T)$ is the time domain.

Exercise 1 1. Consider the scalar problem

$$u_t + au_x = bu \quad x \in (-1, 1) \tag{1}$$

with proper initial conditions and a and b being real constants. **(a)** Propose a discontinuous Galerkin method for solving this problem. **(b)** Prove that the semi-discrete scheme is stable.

2. Consider the scalar PDE

$$u_t + au_x = g(x, t) \tag{2}$$

in , with proper initial conditions and periodic boundary conditions. Here $a > 0$ is a real constant. Write the weak discontinuous Galerkin (DG) formulation for the problem. Pick an appropriate numerical flux and prove that the semi-discrete scheme is stable. (Hint: To prove stability, consider the problem for the error between the computed and the exact solution - known as the error equation).

Consider the system

$$u_t + v_x = 0 \tag{3}$$
$$v_t + u_x = 0 \tag{4}$$

in , subject to periodic boundary conditions. Write the weak DG formulation for the problem. Show that the formulation with the upwind flux is stable.

Consider the ODE system

$$u' = L(u) \ . \tag{5}$$

Suppose there exists a positive constant k_{FE} such that the forward Euler method

$$v^{n+1} = v^n + kL\left(v^n\right)$$

with $0 < k \leq k_{FE}$, applied to (5) satisfies

$$\|v^{n+1}\| \leq \|v^n\| \tag{6}$$

in some norm $\|\cdot\|$. Show that the numerical approximation U of (5) obtained by the following 3rd-order Runga-Kutta method

$$\begin{aligned}
U^{(1)} =& U^n + kL\left(U^n\right) \\
U^{(2)} =& \frac{3}{4}U^n + \frac{1}{4}U^{(1)} + \frac{1}{4}kL\left(U^{(1)}\right) \\
U^{n+1} =& \frac{1}{3}U^n + \frac{2}{3}U^{(2)} + \frac{2}{3}kL\left(U^{(2)}\right)
\end{aligned}$$

satisfies (6), provided $0 < k \leq k_{FE}$.

Solution 1 1.

2. Consider the scalar PDE

$$u_t + au_x = g(x, t) \tag{7}$$

in with proper initial conditions and periodic boundary conditions.

(a) Suppose the approximation u_h is given in the kth element $D^k = \left(x_l^k, x_r^k\right)$ by

$$u_h^k(x,t) = \sum_{n=1}^{N_p} \hat{u}_n^k(t)\ \psi_n^k(x)\ , \tag{8}$$

where ψ_n^k is some basis of the space of polynomials of degree no grater than $N = N_p - 1$. We require that for each $k = 1, \ldots, K$,

$$\left({}_h\ ,\ \psi_j^k\right)_{D^k} = \hat{g}_j^k \qquad\qquad j = 1, \ldots, N_p\ , \tag{9}$$

where ${}_h = \left({}_t + a_x\right) u_h$, $\hat{g}_j^k = \left(g, \psi_j^k\right)_{D^k}$, and

$$(\eta, \psi)_{D^k} = \int_{D^k} \eta\psi \mathrm{d}\, x\ . \tag{10}$$

We use the divergence theorem and substitute the boundary terms by the numerical flux f^*, to get

$$\left({}_t u_h^k\ ,\ \psi_j^k\right)_{D^k} - a\left(u_h^k\ ,\ \left(\psi_j^k\right)'\right)_{D^k} = -\left(f^*\psi_j^k\right)_{x_l^k}^{x_r^k} + \hat{g}_j^k \qquad j = 1, \ldots, N_p\ . \tag{11}$$

Substituting (8) into the last equation, provides

$$\sum_{n=1}^{N_p} \left(\psi_n^k, \psi_j^k\right)_{D^k} \left(\hat{u}_n^k\right)' - a\sum_{n=1}^{N_p} \left(\psi_n^k\ ,\ \left(\psi_j^k\right)'\right)_{D^k} \hat{u}_n^k = -\left(f^*\psi_j^k\right)_{x_l^k}^{x_r^k} + \hat{g}_j^k\ , \tag{12}$$

for $j = 1, \ldots, N_p$. This can be also written as a system,

$${}^{k}\left(\hat{\boldsymbol{u}}_h^k\right)' - a\left({}^{k}\right)^T \hat{\boldsymbol{u}}_h^k = -\left(f^*\boldsymbol{\psi}^k\right)_{x_l^k}^{x_r^k} + \hat{\boldsymbol{g}}^k\ , \tag{13}$$

where

$${}^{k}_{jn} = \left(\psi_n^k, \psi_j^k\right)_{D^k} \qquad {}^{k}_{nj} = \left(\psi_n^k\ ,\ \left(\psi_j^k\right)'\right)_{D^k}\ . \tag{14}$$

The periodic boundary conditions are enforced by requiring $f^*\big|_{x_l^1} = f^*\big|_{x_r^K}$. This is can be obtained by formally defining $u_h^0 = u_h^K$, $u_h^{K+1} = u_h^1$ and then using the same numerical flux at the entire domain (including the boundaries $x = \pm 1$).

(b) We take the dot product of (13) and $\hat{\boldsymbol{u}}_h^k$, and sum over $k = 1, \ldots, K$ to get

$${}_t \left\|u_h\right\|^2 = \sum_{k=1}^{K} \left[a\left(u_r^k\right)^2 - a\left(u_l^k\right)^2 - 2u_r^k f_{k+1/2}^* + 2u_l^k f_{k-1/2}^*\right] + 2\left(g, u_h\right)\ , \tag{15}$$

where, for $k = 1, \ldots, K$, $u_l^k = u_h^k\left(x_l^k, \cdot\right)$, $u_r^k = u_h^k\left(x_r^k, \cdot\right)$, $u_r^0 = u_r^K$, $u_l^{K+1} = u_l^1$, and

$$f_{k+1/2}^* = f^*\left(u_r^k, u_l^{k+1}\right) \quad k = 0, \ldots, K\ . \tag{16}$$

By rearranging the sum we get

$${}_t \left\|u_h\right\|^2 = -\sum_{k=0}^{K} \left(u_r^k - u_l^{k+1}\right)\left(2f_{k+1/2}^* - a\left(u_r^k + u_l^{k+1}\right)\right) + 2\left(g, u_h\right)\ . \tag{17}$$

Substituting the upwind flux $f^*(u_l, u_r) = au_l$ provides

$${}_t \left\|u_h\right\|^2 = -\sum_{k=0}^{K} a\left(u_r^k - u_l^{k+1}\right)^2 + 2\left(g, u_h\right) \leq 2\left(g, u_h\right)\ . \tag{18}$$

Finally, by the Cauchy-Schwarz inequality we get

$$\frac{1}{2}{}_t \left\|u_h\right\|^2 \leq \left(g, u_h\right) \leq \left\|g\right\| \left\|u_h\right\|\ , \tag{19}$$

which leads to the desired bound on the norm of u_h.

3. (a) The system

$$u_t + v_x = 0 \tag{20}$$
$$v_t + u_x = 0 \ , \tag{21}$$

has the form

$$w_t + Aw_x = 0 \tag{22}$$

where A is a symmetric matrix. In the kth element D^k the approximation w_h is given by

$$w_h^k(x,t) = \sum_{n=1}^{N_p} \psi_n^k(x)\hat{w}_n^k(t) \ . \tag{23}$$

We require

$$\int_{D^k} \psi_j^k \left({}_t w_h^k + A_x w_h^k \right) \mathrm{d}\, x = 0 \qquad\qquad k = 1, \ldots, K \ . \tag{24}$$

We integrate by parts, and replace Aw_h^k in the boundary terms by a numerical flux f^* to get

$$\sum_{n=1}^{N_p} \left(\psi_j^k, \psi_n^k\right)_{D^k} \left(\hat{w}_n^k\right)' \ - \sum_{n=1}^{N_p} \left(\left(\psi_j^k\right)', \psi_n^k \right)_{D^k} A\hat{w}_n^k = -\left(\psi_j^k f^*\right)_{x_l^k}^{x_r^k} \ . \tag{25}$$

That is

$$\hat{}^k \left(\hat{\boldsymbol{u}}_n^k\right)' - \left(\hat{}^k\right)^T \hat{\boldsymbol{v}}_h^k = (\boldsymbol{\psi} g^*)_{x_l^k}^{x_r^k} \tag{26}$$
$$\hat{}^k \left(\hat{\boldsymbol{v}}_n^k\right)' - \left(\hat{}^k\right)^T \hat{\boldsymbol{u}}_h^k = (\boldsymbol{\psi} h^*)_{x_l^k}^{x_r^k} \ , \tag{27}$$

where $f^* = (g^*, h^*)^T$.

(b) By taking the dot product of (25) and $\hat{w}_j^k$, and summing over $j = 1, \ldots, N_p$, we get

$$\frac{1}{2} t \left\|w_h^k\right\|_{D^k}^2 - \int_{D^k} w_h^k x \cdot Aw_h^k \ \mathrm{d}\, x = -\left(w_h^k \cdot f^*\right)_{x_l^k}^{x_r^k} \ . \tag{28}$$

Since A is symmetric, this implies

$$t \left\|w_h^k\right\|_{D^k}^2 = \left(w_h^k \cdot \left(Aw_h^k - 2f^*\right)\right)_{x_l^k}^{x_r^k} = w_r^k \cdot \left(Aw_r^k - 2f_{k,k+1}^*\right) - w_l^k \cdot \left(Aw_l^k - 2f_{k-1,k}^*\right) \ . \tag{29}$$

Here $w_r^k = w_h^k\left(x_r^k, \cdot\right)$, and $w_l^k = w_h^k\left(x_l^k, \cdot\right)$. Summing over $k = 1, \ldots, K$ provides

$$t \left\|w_h\right\|^2 = \sum_{k=1}^{K} w_r^k \cdot \left(Aw_r^k - 2f_{k+1/2}^*\right) - \sum_{k=0}^{K-1} w_l^{k+1} \cdot \left(Aw_l^{k+1} - 2f_{k+1/2}^*\right) \tag{30}$$
$$= -\sum_{k=0}^{K} \left[w_l^{k+1} \cdot Aw_l^{k+1} - w_r^k \cdot Aw_r^k - 2\left(w_l^{k+1} - w_r^k\right) \cdot f_{k+1/2}^* \right] \ . \tag{31}$$

Now we substitute the upwind flux

$$f_{k+1/2}^* = \frac{1}{2} \ A\left(w_l^{k+1} + w_r^k\right) - \frac{1}{2} \ |A| \left(w_l^{k+1} - w_r^k\right) \tag{32}$$

into (30) to get

$$t \left\|w_h\right\|^2 = -\sum_{k=1}^{K} \left(w_l^{k+1} - w_r^k\right) \cdot |A| \left(w_l^{k+1} - w_r^k\right) \leq 0 \ . \tag{33}$$

4. Consider the ODE system

$$u' = L(u) \ . \tag{34}$$

Suppose there exists a positive constant k_{FE} such that the forward Euler method

$$v^{n+1} = v^n + kL\left(v^n\right)$$

with $0 < k \leq k_{FE}$, applied to (5) satisfies

$$\|v^{n+1}\| \leq \|v^n\| \tag{35}$$

in some norm $\|\cdot\|$. Suppose U is the numerical approximation of (5) obtained by the following 3rd-order Runga-Kutta method

$$\begin{aligned} U^{(1)} =& U^n + kL\left(U^n\right) \\ U^{(2)} =& \frac{3}{4}U^n + \frac{1}{4}U^{(1)} + \frac{1}{4}kL\left(U^{(1)}\right) \\ U^{n+1} =& \frac{1}{3}U^n + \frac{2}{3}U^{(2)} + \frac{2}{3}kL\left(U^{(2)}\right) \ , \end{aligned}$$

and $0 < k \leq k_{FE}$. Next we show that

$$\|Ui\| \leq \|U^n\| \tag{36}$$

holds for $i = 1, 2$. As $0 < k \leq k_{FE}$, it is clear that $U1$ satisfies (36). This implies

$$\|U2\| \leq \frac{3}{4}\|U^n\| + \frac{1}{4}\left\|U1 + kL\left(U1\right)\right\| \leq \frac{3}{4}\|U^n\| + \frac{1}{4}\|U1\| \tag{37}$$

which by (36) with $i = 1$, implies (36) also for $i = 2$. Finally, we have

$$\|U^{n+1}\| \leq \frac{1}{3}\|U^n\| + \frac{2}{3}\left\|U2 + kL\left(U2\right)\right\| \leq \frac{1}{3}\|U^n\| + \frac{2}{3}\|U2\| \tag{38}$$

which by (36) with $i = 2$, implies the proposition.