Exercise 12: WENO reconstruction

Exercise 1
The WENO reconstruction is based on a convex combination, with coefficients ω_r, of approximations $v^{(r)}_{i+1/2}$ calculated on k different stencils. The coefficients ω_r depend on another set of coefficients d_r and some smoothness indicators β_r.

1. The Smoothness indicators β_r are defined by

$$\beta_r = \sum_{l=1}^{k-1} \Delta x^{2l-1} \int_{x_i-1/2}^{x_{i+1/2}} \left(\frac{\mathrm{d}^l p_r}{\mathrm{d}\, x^l} (x) \right)^2 \mathrm{d}\, x \qquad r = 0, \ldots, k-1 \ . \tag{1}$$

   For $k = 2$ (3rd-order reconstruction), this yields the following expressions:

$$\beta_0 = \left(\overline{U}_{i+1} - \overline{U}_i \right)^2 \ , \qquad \beta_1 = \left(\overline{U}_i - \overline{U}_{i-1} \right)^2 \ . \tag{2}$$

   Verify (2).

2. The coefficients d_r are chosen so that

$$\sum_{r=0}^{k-1} d_r v^{(r)}_{i+1/2} = v\left(x_{i+1/2}\right) + O\left(\Delta x^{2k-1}\right) \ . \tag{3}$$

   For $k = 2, 3$,

$$k = 2: \quad d_0 = \frac{2}{3} \ , \quad d_1 = \frac{1}{3} \tag{4}$$

$$k = 3: \quad d_0 = \frac{3}{10} \ , \quad d_1 = \frac{3}{5} \ , \quad d_2 = \frac{1}{10} \ . \tag{5}$$

   Explain how these values are obtained and verify (4).

3. Write a program that implements WENO reconstruction for both $k = 2, 3$.

4. Write a finite volume code for

$$u_t + a u_x = 0$$

   where the interface values are obtained by using the WENO reconstruction. Plug these values in the Godunov flux, and integrate the semi-discrete scheme using SSP-RK3. Implement the following initial conditions on the domain $[-1, 1]$ with periodic BC

$$v_0(x) = \sin(\pi x) \ , \quad T_f = 5 \tag{6}$$

$$v_0(x) = \begin{cases} 1 & |x| < 0.5 \\ -1 & |x| > 0.5 \end{cases} \ , \quad T_f = 0.5. \tag{7}$$

   Use a CFL of 0.5 to evaluate the time-step.

5. Run the code for $k = 2, 3$ and $a = 1$. Do you recover the expected order of convergence for (6)? Is the solution oscillatory for (7)?

Solution 1 1. We start the calculation by writing

$$P_r(x) = \sum_{j=0}^{k} V\left(x_{i-r+j-1/2}\right) \ell_j \left(\frac{x - x_{i-r-1/2}}{h}\right) , \tag{8}$$

where ℓ_j are the Lagrange polynomials. For $k = 2$, we have

$$\begin{aligned} P_r(x) =& \frac{V\left(x_{i-r-0.5}\right)}{2h^2}\left(x - x_{i-r+1/2}\right)\left(x - x_{i-r+3/2}\right) \\ &- \frac{V\left(x_{i-r+1/2}\right)}{h^2}\left(x - x_{i-r-1/2}\right)\left(x - x_{i-r+3/2}\right) \\ &+ \frac{V\left(x_{i-r+3/2}\right)}{2h^2}\left(x - x_{i-r-1/2}\right)\left(x - x_{i-r+1/2}\right) , \end{aligned}$$

which implies

$$\begin{aligned} \frac{\mathrm{d}\,p_r}{\mathrm{d}\,x}(x) &= \frac{\mathrm{d}^2 P_r}{\mathrm{d}\,x^2}(x) = \frac{1}{h^2}\Big[V\left(x_{i-r+3/2}\right) - 2V\left(x_{i-r+1/2}\right) + V\left(x_{i-r-1/2}\right)\Big] \\ &= \frac{1}{h}\left(\overline{U}_{i-r+1} - \overline{U}_{i-r}\right) \end{aligned}$$

Therefore,

$$\beta_r = h \int_{x_{i-1/2}}^{x_{i+1/2}} \frac{1}{h^2}\left(\overline{U}_{i-r+1} - \overline{U}_{i-r}\right)^2 \mathrm{d}\,x = \left(\overline{U}_{i-r+1} - \overline{U}_{i-r}\right)^2$$

which completes the proof.

2. Suppose v is a smooth function, and for each i, its average on the ith cell $I_i = (x_{i-1/2}, x_{i+1/2})$ is $\overline{v}_i$. Fix some i. The reconstruction of the value of v at (the cell boundary) $x_{i+1/2}$ obtained from the rth stencil of size $k = 2$ is given by

$$v_{i+1/2}^{(r)} = \sum_{j=0}^{1} c_{rj}\overline{v}_{i-r+j} . \tag{9}$$

Explicitly, we have

$$v_{i+1/2}^{(0)} = \frac{1}{2}\overline{v}_i + \frac{1}{2}\overline{v}_{i+1} \quad v_{i+1/2}^{(1)} = -\frac{1}{2}\overline{v}_{i-1} + \frac{3}{2}\overline{v}_i . \tag{10}$$

The weights d_r are chosen so that

$$\sum_{r=0}^{1} d_r v_{i+1/2}^{(r)} = v\left(x_{i+1/2}\right) + O\left(h^3\right) . \tag{11}$$

On the other hand, by (10),

$$\sum_{r=0}^{1} d_r v_{i+1/2}^{(r)} = -\frac{d_1}{2}\overline{v}_{i-1} + \left(\frac{d_0}{2} + \frac{3d_1}{2}\right)\overline{v}_i + \frac{d_0}{2}\overline{v}_{i+1} . \tag{12}$$

This is a 3rd-order approximation to $v\left(x_{i+1/2}\right)$ obtained from $\overline{v}_{i-1+j}$, with $j = 1, 2, 3$. This is simply the reconstruction at $x_{i+1/2}$ obtained from the stencil of size $k = 3$, associated with $r = 1$:

$$\widetilde{v}_{i+1/2}^{(1)} = -\frac{1}{6}\overline{v}_{i-1} + \frac{5}{6}\overline{v}_i + \frac{1}{3}\overline{v}_{i+1} \tag{13}$$

By requiring

$$\widetilde{v}_{i+1/2}^{(1)} = \sum_{r=0}^{1} d_r v_{i+1/2}^{(r)} \tag{14}$$

and comparing coefficients, we get

$$d_0 = \frac{2}{3} , \quad d_1 = \frac{1}{3} . \tag{15}$$

See codes `WENO.m` and `ReconstructWeights.m` attached at the end of the solution manual. In Exercise 11, we had written a slightly different code to evaluate the coefficients c_{rj}. A more compact algorithm is used for this exercise (also available in the Software folder uploaded on the moodle platform).

See codes attached at the end of the solution manual.

If you look at the plots generated by the code, we obtain third-order convergence for k=2,3 with a smooth initial condition. Even though we use a fifth-order discretization in space, we are limited by the fact the time integration is only third-order accurate, and $\Delta t \sim h$. Furthermore, the solutions are non-oscillatory with the discontinuous initial condition.