

Exercise Set 11: ENO reconstruction

Exercise 1

Consider the linear advection equation

$$v_t + av_x = 0 . \quad (1)$$

In previous exercises, we had observed that a fixed stencil approach for reconstruction is not suitable for obtaining stable non-oscillatory solutions. In the following exercises you are asked to write a number of programs which calculate different parts of the ENO reconstruction. Assume the grid is uniform.

1. Write a program that takes as an input a stencil-size k , and a grid function v , and finds for each cell an appropriate stencil of size k . For example, for the i th cell $I_i = (x_{i-1/2}, x_{i+1/2})$, the function should return the output r_i , if the chosen stencil is $x_{i-r_i}, \dots, x_{i-r_i+k-1}$. Here, the value v_i of v at x_i is interpreted as average of the approximated function v in I_i . Notice that since the grid is assumed to be uniform, you can make your program slightly more efficient by computing the *undivided differences*,

$$V\langle x_{i-1/2}, x_{i+1/2} \rangle := V(x_{i-1/2}, x_{i+1/2}) = v_i \quad (2)$$

$$V\langle x_{i-1/2}, \dots, x_{i+j+1/2} \rangle := V\langle x_{i+1/2}, \dots, x_{i+j+1/2} \rangle - V\langle x_{i-1/2}, \dots, x_{i+j-1/2} \rangle \quad (3)$$

instead of the divided differences.

2. Write a program that calculates the coefficients

$$c_{rj} = \sum_{m=j+1}^k \frac{1}{\prod_{\substack{l=0 \\ l \neq m}}^k (m-l)} \sum_{\substack{l=0 \\ l \neq m}}^k \prod_{\substack{q=0 \\ q \neq m, l}}^k (r-q+1) \quad j = 0, \dots, k-1, \quad r = -1, 0, \dots, k-1 . \quad (4)$$

Explain how this expression is derived. Check that this formula generates the coefficients obtained in Exercise 10 for $k = 3$.

3. Write a program that implements the ENO reconstruction. The program should take a grid function v , and a stencil-size k , and return the values

$$v_{i+1/2}^- = \sum_{j=0}^{k-1} c_{r_i, j} v_{i-r_i+j}^n , \quad v_{i-1/2}^+ = \sum_{j=0}^{k-1} c_{r_i-1, j} v_{i-r_i+j}^n . \quad (5)$$

You may assume the function is periodic.

4. Write a finite volume code for (1), where the interface values can be obtained by using the ENO reconstruction. Plug these values in the Godunov flux, and integrate the semi-discrete scheme using SSP-RK3. Implement the following initial conditions on the domain $[-1, 1]$ with periodic BC

$$v_0(x) = \sin(\pi x) , \quad T_f = 5 \quad (6)$$

$$v_0(x) = \begin{cases} 1 & |x| < 0.5 \\ -1 & |x| > 0.5 \end{cases} , \quad T_f = 0.5. \quad (7)$$

Use a CFL of 0.5 to evaluate the time-step.

5. Run the code for $k = 3$ and $a = 1$. Do you recover 3rd-order convergence for (6)? Is the solution oscillatory for (7)?

- Solution 1**
1. See the Matlab code attached at the end of this solution manual. The stencil selection depends on the index shift r , which is evaluated by the function `find_shift(U,k)`.
 2. The coefficients c_{rj} are computed once at the beginning of the solver by the function `eval_crj(k)`, assuming a uniform grid.
 3. Once the shift r is evaluate for each cell, the solution at the left and right interfaces of the cell are computed in the function `eno_recon`.
 4. Note that the code is capable of implementing both periodic and open boundary conditions, for arbitrary order ENO. The order can be set via the variable k .
 5. Third-order convergence is obtained for the smooth initial condition. Moreover, the oscillations observed with the discontinuous initial data for fixed central reconstruction in Exercise 9, are no longer visible.