Exercise Set 10: High-order schemes and stencil selection

Exercise 1
Consider the linear advection equation

$$u_t + au_x = 0 \ . \tag{1}$$

We can construct a high-order scheme for (1) by suitably reconstructing the interface values (which is used to evaluate the numerical flux), followed by a Runge-Kutta integration in time. In each cell i, a quadratic polynomial can be obtained using the cell-average values, by choosing one of the following stencils

$$S_r = \{x_{i-r} \ , \ x_{i+1-r} \ , \ x_{i+2-r}\}, \quad r = 0, 1, 2$$

where r represents the number of cells to the left of cell i in the stencil S_r. For a fixed r, the left and right interface values can be expressed as

$$u^-_{i+\frac{1}{2}} = \sum_{j=0}^{2} c_{rj} \overline{u}_{i-r+j} \ , \qquad u^+_{i-\frac{1}{2}} = \sum_{j=0}^{2} \tilde{c}_{rj} \overline{u}_{i-r+j} \tag{2}$$

each of which are third-order accurate approximations.

1. Find the coefficients c_{rj} and $\tilde{c}_{rj}$ for $r, j = 0, 1, 2$. (You could use two methods to obtain these: i) differentiating the interpolating polynomial for the primitive of u, or ii) directly using Taylor series expansions and trying to satisfy order constraints.)

2. Write a finite volume code for (1), where the interface values can be obtained by using either of the three stencils. Plug these values in the Godunov flux, and integrate the semi-discrete scheme using SSP-RK3. Implement the following initial conditions on the domain $[-1, 1]$

$$u_0(x) = \sin(\pi x) \ , \quad T_f = 5, \quad \text{with periodic BC} \tag{3}$$

$$u_0(x) = \begin{cases} 1 & x < 0 \\ -1 & x > 0 \end{cases} \ , \quad T_f = 0.5, \quad \text{with open BC.} \tag{4}$$

   Use a CFL of 0.2 to evaluate the time-step.

3. Run the code for $r = 0, 1, 2$ and $a = 1$. What do you observe with each type of stencil? Do you recover 3rd-order convergence for (3)? How do the results change if you choose $a = -1$ instead?

Solution 1 1. The interface values are given by

$$u^-_{i+\frac{1}{2}} = \sum_{j=0}^{2} c_{r,j} \overline{u}_{i-r+j} \ , \qquad u^+_{i-\frac{1}{2}} = \sum_{j=0}^{2} \tilde{c}_{r,j} \overline{u}_{i-r+j} \tag{5}$$

where $\tilde{c}_{r,j} = c_{r-1,j}$. The coefficients are given in Table 1. Note that we also define the values for r=-1, since it is needed to define $\tilde{c}_{r,j}$ when r=0.

2. See the Matlab code attached at the end of this solution manual.

j / r	0	1	2
-1	$\frac{11}{6}$	$-\frac{7}{6}$	$\frac{1}{3}$
0	$\frac{1}{3}$	$\frac{5}{6}$	$-\frac{1}{6}$
1	$-\frac{1}{6}$	$\frac{5}{6}$	$\frac{1}{3}$
2	$\frac{1}{3}$	$-\frac{7}{6}$	$\frac{11}{6}$

Table 1: Table of coefficients.

3. See figures generated by the Matlab code.

   The solutions evaluated with r=0,2 are not stable, and blow up quickly with mesh refinement. However, the solutions are stable with r=1, which corresponds to choosing the central stencil for each cell. Third-order accuracy is observed for the smooth initial condition with r=1, which drops to below first-order for the discontinuous data. Furthermore, small Gibbs oscillations appear near the discontinuity.

   The results from this exercise indicate that simply using a high-order reconstruction may not ensure stability. One can obtain stable solutions by adaptively choosing the stencil for reconstruction in each cell. This is the strategy used in the so called *essentially non-oscillatory* (ENO) reconstruction method.

Strong Stability Preserving Runge-Kutta scheme (SSP-RK3)

The algorithm for SSP-RK3 to solve an ODE of the form

$$\frac{\mathrm{d}\mathbf{q}_i}{\mathrm{d}t} = \mathbf{L}(\mathbf{q}, t)$$

is given by

$$\begin{aligned}
\mathbf{q}^{(1)} &= \mathbf{q}^n + k\mathbf{L}\left(\mathbf{q}^n, t^n\right)\\
\mathbf{q}^{(2)} &= \frac{3}{4}\mathbf{q}^n + \frac{1}{4}\left(\mathbf{q}^{(1)} + k\mathbf{L}\left(\mathbf{q}^{(1)}, t^n + k\right)\right)\\
\mathbf{q}^{(3)} &= \frac{1}{3}\mathbf{q}^n + \frac{2}{3}\left(\mathbf{q}^{(2)} + k\mathbf{L}\left(\mathbf{q}^{(2)}, t^n + k/2\right)\right)\\
\mathbf{q}^{n+1} &= \mathbf{q}^{(3)}.
\end{aligned}$$