Tutorial Project N°1

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Numerical Methods for Conservation Laws
Autumn Term 2023

December 6, 2023

Project N°1: The Finite Difference Method

One-dimensional Shallow Water Equation

Find $\mathbf{q} = (h, m)^\top$ such that

$$\frac{\partial \mathbf{q}}{\partial t} + \frac{\partial \mathbf{f}(\mathbf{q})}{\partial x} = \mathbf{S}(x, t)$$

- $h(x, t)$: Depth of height of the water.
- $m(x, t)$: Discharge, measure the flow rate of the fluid at (x, t)
- g: Acceleration of gravity (We assume g=1).
- S(x,t): Source term.
- Physical domain $\Omega = (0, 2)$.
- Flux

$$\mathbf{f}(\mathbf{q}) = \begin{pmatrix} m \\ \frac{m^2}{h} + \frac{1}{2} g h^2 \end{pmatrix}$$

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Problem N°1

- Discretize Ω with N cells.
- Implement the Lax-Friedrichs method to solve the shallow water equations using the FDM.
- **Boundary conditions**
  - **Periodic** Boundary conditions, i.e. $\mathbf{q}_0 = \mathbf{q}_N$ and $\mathbf{q}_{N+1} = \mathbf{q}_1$.
  - **Open** Boundary conditions, i.e. $\mathbf{q}_0 = \mathbf{q}_1$ and $\mathbf{q}_{N+1} = \mathbf{q}_N$.
- Test your code with

$$h(x,0) = 1 + 0.5\sin(\pi x) \quad \text{and} \quad m(x,0) = uh_0(x),$$

  with $u = 0.25$.
- **Exact Solution is Given!**

$$h(x,t) = h_0(x-t) \quad \text{and} \quad m(x,t) = uh(x,t).$$

- **CFL Condition**

$$k = \mathrm{CFL}\frac{\Delta x}{\max_i(|u_i| + \sqrt{gh_i})}$$

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- Set $\mathbf{q}_j^n = (h_j^n, m_j^n)^\top$.
- Conservative Finite Differences

$$\mathbf{q}_j^{n+1} = \mathbf{q}_j^n - \frac{k}{\Delta x}\left(\mathbf{F}_{j+\frac{1}{2}}^n - \mathbf{F}_{j-\frac{1}{2}}^n\right).$$

  with $\mathbf{F}_{j+\frac{1}{2}}^n = F(\mathbf{q}_j^n, \mathbf{q}_{j+1}^n)$.

- Recall that in this problem

$$\mathbf{f}(\mathbf{q}) = \begin{pmatrix} m \\ \frac{m^2}{h} + \frac{1}{2}gh^2 \end{pmatrix}$$

- Lax-Friedrichs Flux

$$F_{\mathrm{LF}}(\mathbf{u}, \mathbf{v}) = \frac{\mathbf{f}(\mathbf{u}) + \mathbf{f}(\mathbf{v})}{2} - \frac{1}{2}\,\frac{\Delta x}{k}\,(\mathbf{v} - \mathbf{u})$$

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Problem 2

- Set $\mathbf{S} = 0$.
- **Initial Conditions**

$$h(x,0) = 1 - 0.1\sin(\pi x), \qquad m(x,0) = 0$$

  and

$$h(x,0) = 1 - 0.2\sin(2\pi x), \qquad m(x,0) = 0.5$$

- **Periodic Boundary Conditions**
- Plot the solution at $T = 0.5$.
- Comment on the regularity of the solution and the performance of the scheme.
- For each initial condition, measure the error at $T = 0.5$. as a function of Δx and plot the results in log-log scale.
- Reference Solution: Use a very fine mesh.

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Problem 3

- For $\mathbf{S} = 0$, implement the following initial condition

$$h(x,0) = 1 \ , \qquad m(x,0) = \begin{cases} -0.5 & x < 1 \\ 0 & x > 1 \end{cases} \ ,$$

  with open boundary conditions.

- Obtain a reference solution using the Lax-Friedrichs flux.
- Plot the solutions with the Lax-Friedrichs scheme at time $T = 0.5$ with varying mesh size, and compare them with the reference solution.
- Comment on the type of waves that can be seen in the solution.
- Repeat the same experiment with the Lax-Wendroff scheme (Corrected in the homepage).

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Final Report

- You can work in groups of two, or individually.
- You have to submit a final written report, hopefully typed in Latex.
- Key points to include
  - Description of the FDM for the shallow water problem.
  - Report numerical experiments.
  - Discuss the obtained numerical results and answer the questions.
  - Include Codes: Append the codes as listing in the report and submit the codes themselves.
- Further questions can be discussed on the exercise session.

Remember: The final grade will be the best of the following three options

- 100% final exam (presumably an oral exam as in last years)
- 90% final exam and 10% best project
- 80% final exam and 10% each project

Best of Success!