Exercise Class #8
Numerical Methods for Conservation Laws

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Exercise Set #8 - Exercise 1

Exercise 1

- Discuss qualitatively the derivation of *Godunov's method*.
- Sketch each step in the solution process.
- Which part of the algorithm can make its implementation particularly difficult?

Exercise Set #8 - Exercise 1

Exercise 1

- Godunov's method can be outlined in two steps.
- Suppose we have an approximation v^n of the solution u at t_n.

  (i) Define $\widetilde{u}^n(x,t)$ for all x and $t_n < t < t_{n+1} = t_n + k$ as the exact solution to the conservation law, satisfying the initial condition

  $$\widetilde{u}^n(x,t_n) = v_j^n \quad x \in (x_{j-1/2}, x_{j+1/2}) \quad \forall j \ . \tag{1}$$

  (ii) Average the resulting function $\widetilde{u}^n(x,t_{n+1})$ over each cell $(x_{j-1/2}, x_{j+1/2})$ to obtain the approximation

  $$v_j^{n+1} = \frac{1}{h}\int_{x_{j-1/2}}^{x_{j+1/2}} \tilde{u}^n(x,t_{n+1})\,\mathrm{d}\,x \tag{2}$$

  at t_{n+1}.

Now this procedure can be repeated to advance to the next time-step.

Exercise Set #7 - Exercise 1

Exercise 1

- In Step (i), we need to solve an exact Riemann problem at each cell-interface, over a small time interval (t_n, t_{n+1}). Since $\widetilde{u}^n$ is a solution to the conservation law, (2) yields

$$v_j^{n+1} = \frac{1}{h}\int_{x_{j-1/2}}^{x_{j+1/2}} \tilde{u}^n\left(x, t_n\right) \mathrm{d}\, x \tag{3}$$

$$- \frac{1}{h}\left(\int_{t_n}^{t_n+k} f\left(\tilde{u}^n\left(x_{j-1/2}, t\right)\right) \mathrm{d}\, t - \int_{t_n}^{t_n+k} f\left(\tilde{u}^n\left(x_{j-1/2}, t\right)\right) \mathrm{d}\, t\right) . \tag{4}$$

  First notice that $v_j^n = \frac{1}{h}\int_{x_{j-1/2}}^{x_{j+1/2}} \tilde{u}^n\left(x, t_n\right) \mathrm{d}\, x$, since $\widetilde{u}^n$ satisfies (1).

- $\widetilde{u}^n\left(x_{j-1/2}, \cdot\right)$ is constant over a small time window. Why?

- Unfortunately, evaluating the intermediate can be very expensive, and at times impossible.

- This motivates the need to construct *approximate Riemann solvers*.

Exercise Set #7 - Exercise 2

Exercise 2

- Consider the scalar conservation law

$$u_t + f(u)_x = 0 \ , \tag{5}$$

and initial condition

$$u(x,0) = \begin{cases} u_l & x < 0 \\ u_r & 0 < x \end{cases} \ , \tag{6}$$

where the flux f is convex ($f'' > 0$).

Exercise Set #7 - Exercise 2

Exercise 2: Part I

- Godunov's method relies on finding the intermediate state $u^* = u^*(u_l, u_r)$ for which $u(0, t) = u^*$, for $t > 0$.
- Show that this intermediate state is given by the following:

1. $f'(u_l), f'(u_r) \geqslant 0 \implies u^* = u_l$
2. $f'(u_l), f'(u_r) \leqslant 0 \implies u^* = u_r$
3. $f'(u_l) \geqslant 0 \geqslant f'(u_r) \implies u^* = \begin{cases} u_l & s > 0 \\ u_r & s < 0 \end{cases}, \ s = \dfrac{f(u_r) - f(u_r)}{u_r - u_l}$
4. $f'(u_l) < 0 < f'(u_r) \implies u^* = u_m$,

where u_m is the solution to $f'(u_m) = 0$.

Exercise Sheet #8 - Exercise 2

Theorem 4.12. *Let $q \in C^2(\mathbb{R})$ be strictly convex and $q'' \geq h > 0$ or strictly concave and $q'' \leq -h < 0$.*

a) *If $q'' > h$ and $u_- > u_+$ or $q'' < h$ and $u_- < u_+$, the unique entropy solution is given by the shock wave*

$$u(x,t) = \begin{cases} u_+ & x > \sigma(u_-, u_+)\, t \\ u_- & x < \sigma(u_-, u_+)\, t \end{cases} \tag{4.65}$$

where

$$\sigma(u_-, u_+) = \frac{q(u_+) - q(u_-)}{u_+ - u_-}.$$

b) *If $q'' > h$ and $u_- < u_+$ or $q'' < -h$ and $u_- > u_+$, the unique entropy solution is given by the rarefaction wave*

$$u(x,t) = \begin{cases} u_- & x < q'(u_-)\, t \\ r\left(\frac{x}{t}\right) & q'(u_-)\, t < x < q'(u_+)\, t \\ u_+ & x > q'(u_+)\, t \end{cases}$$

where $r = (q')^{-1}$ is the inverse function of q'.

Exercise Set #8 - Exercise 2 - Hints

Exercise 2 - Hints

- Suppose $f'(u_l), f'(u_r) \geqslant 0$.
- If $u_l > u_r$, the entropy solution is a shock moving at speed given by the RH condition

$$s = \frac{f(u_l) - f(u_r)}{u_l - u_r} \tag{7}$$

  and $f'(u_l) > s > f'(u_r)$ according to the entropy condition.
- This implies that the shock speed is positive, and thus we have $u^* = u_l$.
- If $u_l \leqslant u_r$, the entropy solution is a rarefaction wave.
- Since $f'(u_l) > 0$, the left front of the wave moves to the right, and thus $u^* = u_l$.

Exercise Set #8 - Exercise 2

Exercise 2: Part II

- Use (a) to show that Godunov's flux is given by

$$F(u_l, u_r) = \begin{cases} \min\limits_{u_l \leqslant u \leqslant u_r} f(u) & u_l \leqslant u_r \\ \max\limits_{u_r \leqslant u \leqslant u_l} f(u) & u_l > u_r \end{cases}. \tag{8}$$

- Show that Godunov's flux (8) is monotone.

Exercise Set #7 - Exercise 2

Exercise 2: Part III

- Show that Godunov's flux (8) is monotone.
- To show that Godunov's flux, given by (8), is monotone, we show that it is non-decreasing in its first argument and non-increasing in its second argument.
- If $u_l < u_r$ and $\epsilon > 0$ is small enough, then

$$F\left(u_l+\epsilon, u_r\right) = \min_{u_l+\epsilon \leqslant u \leqslant u_r} f\left(u\right) \geqslant \min_{u_l \leqslant u \leqslant u_r} f\left(u\right) = F\left(u_l, u_r\right)\ , \tag{9}$$

and if $u_l \geqslant u_r$, then

$$F\left(u_l+\epsilon, u_r\right) = \max_{u_r \leqslant u \leqslant u_l+\epsilon} f\left(u\right) \geqslant \max_{u_r \leqslant u \leqslant u_l} f\left(u\right) = F\left(u_l, u_r\right)\ . \tag{10}$$

Similarly, one can show that F is non-increasing in its second argument.

Exercise Set #8 - Exercise 3

Exercise 3

- The purpose of this exercise is to illustrate the *Lax-Wendroff Theorem*.
- Convergence is not a conclusion of the Lax-Wendroff theorem.
- Also recall that in general weak solutions are not unique, so the theorem does not guarantee the limit is the correct entropy solution.
- Consider a conservative method

$$v_j^{n+1} = v_j^n - \frac{k}{h}\left(F\left(v_j^n, v_{j+1}^n\right) - F\left(v_{j-1}^n, v_j^n\right)\right) \tag{11}$$

where the numerical flux F is given by

$$F(v, w) = \begin{cases} f(v) & \frac{f(v)-f(w)}{v-w} \geqslant 0 \\ f(w) & \frac{f(v)-f(w)}{v-w} < 0 \end{cases}. \tag{12}$$

Exercise Set #8 - Exercise 3

Exercise 3

- Construct the entropy solution to the following initial value problem

$$u_t + \left(\frac{1}{2}u^2\right)_x = 0 \quad u(x,0) = \begin{cases} -1 & x < 1 \\ 1 & x > 1 \end{cases}. \tag{13}$$

- Fix $k/h = 0.5$, and implement the above method to (13), in $x \in (0,2)$, $0 < t \leqslant 0.25$, with the initial data discretized using cell averages.
- On the boundaries, set $u(0,t) = -1$, and $u(2,t) = 1$.
- Run the computations by choosing i) $h_l = \frac{2}{l}$, ii)$h_l = \frac{2}{2l}$, iii) $h_l = \frac{2}{2l+1}$, for $l \in \mathbb{N}$.
- What can you deduce from your results regarding the each of the three sequences of numerical solutions obtained?
- Explain your results and conclude how they fit with the Lax-Wendroff theorem.