# Exercise Class #7
# Numerical Methods for Conservation Laws

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Friday 10th of November, 2023

# Exercise Set #7 - Harten's Lemma

### Exercise 1: Non-conservative scheme

- Consider the conservation law

$$u_t + f(u)_x = 0, \tag{1}$$

and the numerical scheme

$$u_i^{n+1} = G(u_{i-k}^n, \ldots, u_{i+k}^n). \tag{2}$$

# Exercise Set #7 - Harten's Lemma

## Exercise 1: Non-conservative scheme

- We say that the scheme (2) can be put in *incremental form* if there exists two incremental coefficients

$$C_{i+\frac{1}{2}} = C(u^n_{i-k+1}, \dots, u^n_{i+k}) \quad \text{and} \quad D_{i+\frac{1}{2}} = D(u^n_{i-k+1}, \dots, u^n_{i+k})$$

  which can be used to re-write the scheme as

$$u_i^{n+1} = u_i^n - C_{i-\frac{1}{2}} \Delta^- u_i^n + D_{i+\frac{1}{2}} \Delta^+ u_i^n, \tag{3}$$

  where $\Delta^+ u_i = u_{i+1} - u_i$ and $\Delta^- u_i = u_i - u_{i-1}$.

- Harten's lemma states that a scheme written in incremental form is TVD if
  - ✓ $C_{i+\frac{1}{2}} \geqslant 0$,
  - ✓ $D_{i+\frac{1}{2}} \geqslant 0$, and
  - ✓ $C_{i+\frac{1}{2}} + D_{i+\frac{1}{2}} \leqslant 1$.

# Exercise Set #7 - Exercise 1

## Exercise 1

- Prove that any 3-point consistent, conservative scheme with numerical flux $F_{i+\frac{1}{2}}$ admits an incremental form with coefficients

$$C_{i+\frac{1}{2}} = \frac{k}{h}\left(\frac{f(u_{i+1}) - F_{i+\frac{1}{2}}}{\Delta^+ u_i}\right), \qquad D_{i+\frac{1}{2}} = \frac{k}{h}\left(\frac{f(u_i) - F_{i+\frac{1}{2}}}{\Delta^+ u_i}\right).$$

## Hints

- Consider the scheme

$$u_i^{n+1} = u_i^n - \frac{k}{h}\left[F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}}\right]. \tag{4}$$

- The RHS of (4) can be re-written as

$$\begin{aligned} RHS &= u_i^n - \frac{k}{h}\left[F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}}\right] \\ &= u_i^n - \frac{k}{h}\left[F_{i+\frac{1}{2}} - f(u_i) + f(u_i) - F_{i-\frac{1}{2}}\right] \\ &= \ldots. \end{aligned}$$

# Exercise Set #7 - Exercise 2

## Exercise 2

- Consider a conservative scheme with
  - ✓ Lax-Friedrich flux:
  $$F^{LF}(u,v) = \frac{1}{2}\left(f(u) + f(v) - \frac{h}{k}(v-u)\right),$$
  - ✓ Local Lax-Friedrich/ Rusanov flux:
  $$F^{LLF}(u,v) = \frac{1}{2}\left(f(u) + f(v) - \alpha(v-u)\right), \quad \alpha = \max_u |f'(u)|,$$
  - ✓ Lax-Wendroff flux:
  $$F^{LW}(u,v) = \frac{1}{2}\left(f(u) + f(v) - \frac{k}{h}f'\left(\frac{u+v}{2}\right)(f(v) - f(u))\right),$$
  - ✓ Roe flux:
  $$F^{Roe}(u,v) = \frac{1}{2}\left(f(u) + f(v) - \alpha(v-u)\right), \quad \alpha = \left|\frac{f(v) - f(u)}{v-u}\right|.$$

# Exercise Set #7 - Exercise 2

### Exercise 2

- Find the incremental coefficients for each flux.
- Check whether all three conditions of Harten's lemma are satisfied with each flux.
- Can you say whether the numerical solution obtained with a TVD scheme is guaranteed to converge to an entropy solution? What happens with the Roe flux?

# Exercise Set #7 - Exercise 2 - Hints

## Exercise 2 - Hints

- Verify that the four fluxes can be written in the form

$$F(u,v) = \frac{1}{2}\left(f(u) + f(v) - Q(u,v)(v-u)\right)$$

where

$$\begin{aligned}
Q^{LF}(u,v) &= \frac{h}{k}\ , \\
Q^{LLF}(u,v) &= \alpha = \max_u |f'(u)|\ , \\
Q^{LW}(u,v) &= \frac{k}{h} f'\left(\frac{u+v}{2}\right)\left(\frac{f(v)-f(u)}{v-u}\right)\ , \\
Q^{Roe}(u,v) &= \left|\frac{f(v)-f(u)}{v-u}\right|\ .
\end{aligned}$$

- Find expressions for incremental coefficients.
- Find conditions for the incremental coefficients to satisfy the hypothesis of Harten's Lemma.
- TVD scheme under which conditions? Can you say whether the numerical solution obtained with a TVD scheme is guaranteed to converge to an entropy solution?

# Exercise Set #7 - Exercise 3

## Exercise 3

- Let $f(u) = cu$. Consider a scheme with the hybrid flux,

$$F(u, v) = \theta F^{LW} + (1 - \theta) F^{LF}, \quad 0 \leqslant \theta \leqslant 1,$$

  which is nothing but a convex combination of the Lax-Friedrich and Lax-Wendroff fluxes.

- Assuming the usual CFL condition, can you find a $\theta$ that will lead to a TVD scheme?

# Exercise Set #7 - Exercise 3 - Hints

## Exercise 3 - Hints

- Let us consider the hybrid flux

$$F^{\theta}(u,v) = \theta F^{LW} + (1-\theta)F^{LF}, \quad 0 \leqslant \theta \leqslant 1,$$

which can also be written as

$$F^{\theta}(u,v) = \frac{1}{2}\left(f(u) + f(v) - Q^{\theta}(u,v)(v-u)\right) \ ,$$

where

$$Q^{\theta}(u,v) = \theta Q^{LW}(u,v) + (1-\theta)Q^{LF}(u,v) \ .$$

Assuming the flux to be linear, i.e., $f(u) = cu$, we get the simplified expression

$$Q^{\theta}(u,v) = \theta \frac{k}{h}c^2 + (1-\theta)\frac{h}{k} = \theta\left[\left(\frac{k}{h}c\right)^2 - 1\right]\frac{h}{k} + \frac{h}{k} \ , \tag{5}$$

- Use conditions from the previous exercise to find the value of $\theta$.