Exercise Class #6
Numerical Methods for Conservation Laws

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Exercise Class #6

Today's Topics: Exercise Set #6

- Conservative Method
- Monotone Methods
- Convergence of FD Schemes
- Discrete Entropy Conditions

Exercise Set #6 - Exercise 1

Exercise 1: Non-conservative scheme

- Consider the non-conservative scheme for Burgers' equation:

$$U_j^{n+1} = U_j^n - \frac{k}{h} U_j^n \left(U_j^n - U_{j-1}^n \right).$$

- Show that this scheme maps non-negative monotonically increasing sequences onto non-negative monotonically increasing sequences.

Hints

- The scheme can be expressed in the following form: $U_j^{n+1} = G(U_{j-1}^n, U_j^n)$ with

$$\begin{aligned} G(V, W) &= W - \frac{k}{h} W (W - V) \\ &= W \left(1 - \frac{k}{h} W \right) + \frac{k}{h} VW \end{aligned}$$

Exercise Set #6 - Exercise 2

Exercise 2: Monotonicity of FTCS

- What is the flux of the FTCS scheme for the transport equation?
- Relate it to other fluxes that you know.
- Determine whether the FTCS scheme is monotone.

Exercise Set #6 - Exercise 2 - Hints

Exercise 2 - Hints

- The FTCS (Forward in Time, Central in Space) uses the following approximations

$$\partial_t u(x_j, t^n) \approx \frac{U_j^{n+1} - U_j^n}{k} \quad \text{and} \quad \partial_x u(x_j, t^n) \approx \frac{U_{j+1}^n - U_{j-1}^n}{2h}. \tag{1}$$

- The FTCS scheme reads

$$U_j^{n+1} = U_j^n - \frac{ak}{2h}\left(U_{j+1}^n - U_{j-1}^n\right). \tag{2}$$

  The flux in this case is $F(U) = aU$.

- The FTCS scheme can be expressed in the following form: $U_j^{n+1} = G(U_{j-1}^n, U_j^n, U_{j+1}^n)$ with

$$G(U, V, W) = V - \frac{ak}{2h}(W - U). \tag{3}$$

- One can readily conclude that the scheme is not monotone.

Exercise Set #6 - Exercise 3

Exercise 3: Order of Convergence

The FTFS (Forward in Time, Forward in Space) uses the following approximations

$$\partial_t u(x_j, t^n) \approx \frac{U_j^{n+1} - U_j^n}{k} + \mathcal{O}(k) \quad \text{and} \quad \partial_x u(x_j, t^n) \approx \frac{U_{j+1}^n - U_j^n}{h} + \mathcal{O}(h^2). \tag{4}$$

The FTCS (Forward in Time, Central in Space) uses the following approximations

$$\partial_t u(x_j, t^n) \approx \frac{U_j^{n+1} - U_j^n}{k} + \mathcal{O}(k) \quad \text{and} \quad \partial_x u(x_j, t^n) \approx \frac{U_{j+1}^n - U_{j-1}^n}{2h} + \mathcal{O}(h^2). \tag{5}$$

Exercise Set #6 - Exercise 4:

Exercise 4: Linear Scheme

- Under what conditions is a linear scheme $F(U, V) = w_1 U + w_2 V$ a monotone scheme?
- Describe the discrete Kruzkov entropy entropy-flux pairs.

Hints

To show that the flux $F(U, V) = w_1 U + w_2 V$ leads to a monotone scheme, we need to show that

$$\lambda \partial_1 F(U_{j-1}^n, U_j^n) \geqslant 0 \tag{6}$$

$$-\lambda \partial_2 F(U_j^n, U_{j+1}^n) \geqslant 0 \tag{7}$$

$$1 - \lambda(\partial_1 F(U_j^n, U_{j+1}^n) - \partial_2 F(U_{j-1}^n, U_j^n)) \geqslant 0 \tag{8}$$

where $\lambda = k/h$.

Exercise Set #6 - Exercise 5

Exercise 5: Roe Flux

- Show that the Roe flux can be written as

$$F(U, V) = \frac{f(V) + f(U)}{2} + \frac{1}{2}\,\mathrm{sgn}(V - U)|f(V) - f(U)|. \tag{9}$$

- Either show that the Roe flux is monotonicity-preserving or give a counter example.

Hints

- The Roe flux is defined as

$$F(u, v) = \begin{cases} f(u), & s \geqslant 0, \\ f(v), & s \leqslant 0, \end{cases} \tag{10}$$

where $s = \frac{f(u) - f(v)}{u - v}$.