Exercise Class #5
Numerical Methods for Conservation Laws

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Exercise Class #5

Today's Topics: Exercise Set #5

- Conservative Schemes
- Monotone Schemes
- Lax-Friedrichs Method
- Implementation

Exercise Set #5 - Exercise 1

Exercise 1: Conservative Methods

- Answer the following questions:
  - ✓ When is a numerical method said to be conservative?
  - ✓ What is the benefit of using a conservative numerical method?
  - ✓ For a given conservation law and a conservative scheme, are we guaranteed that the weak solution obtained satisfies the entropy condition?

Exercise Set #5 - Exercise 1: Hints

Exercise 1: Conservative Methos

- If it can be written in the form

$$v_j^{n+1} = v_j^n - \frac{k}{h}\left[F_{j+\frac{1}{2}}^n - F_{j-\frac{1}{2}}^n\right] \tag{1}$$

  where $F_{j+\frac{1}{2}}^n = F(U_{j-p}^n, \ldots, U_{j+q}^n)$ is the numerical flux at the cell-interface $x_{j+\frac{1}{2}}$.

- Lax-Wendroff theorem.

- No, not in general. Provide a counter-example

Exercise Set #5 - Exercise 2

Exercise 2: Engquist-Osher flux

- Consider the conservative scheme with the *Engquist-Osher* flux

$$F^{EO}(u,v) = \frac{1}{2}\left(f(u) + f(v) - \int_u^v |f'(\xi)|\,d\xi \right). \tag{2}$$

  - ✓ Show that the numerical flux leads to a monotone scheme under suitable CFL conditions.
  - ✓ Assuming f is convex with a single minima at ω, show that the Engquist-Osher flux reduces to

$$F^{EO}(u,v) = f(\max(u,\omega)) + f(\min(v,\omega)) - f(\omega). \tag{3}$$

Exercise Set #5 - Exercise 2: Hints

Exercise 2: Engquist-Osher flux

- To show that the Engquist-Osher flux (EO flux) leads to a monotone scheme, we need to show that

$$\lambda \partial_1 F(U_{j-1}^n, U_j^n) \geqslant 0 \tag{4}$$

$$-\lambda \partial_2 F(U_j^n, U_{j+1}^n) \geqslant 0 \tag{5}$$

$$1 - \lambda(\partial_1 F(U_j^n, U_{j+1}^n) - \partial_2 F(U_{j-1}^n, U_j^n)) \geqslant 0 \tag{6}$$

  where $\lambda = k/h$.

- Why?

Exercise Set #5 - Exercise 2: Hints

Exercise 2: Engquist-Osher flux

- Assume f is convex with a minima at ω.
- Then we have $f'(u) < 0$ if $u < \omega$ and $f'(u) > 0$ if $u > \omega$.
- Considering 4 possible scenarios, we have

$$\int_u^v |f'(\xi)| d\xi = \begin{cases} f(v) - f(u), & \text{if } u, v > \omega \\ f(u) - f(v), & \text{if } u, v < \omega \\ f(u) + f(v) - 2f(\omega), & \text{if } u < \omega < v \\ 2f(\omega) - f(u) - f(u), & \text{if } v < \omega < u \end{cases}. \tag{7}$$

Thus, the numerical flux becomes....(You should complete this part).

Exercise Set #4 - Exercise 3

Exercise 3: Finite Differences for Burger's Equation

- In the previous exercises, we have seen that difficulties can arise when trying to approximate solution for linear problem.
- Additional issues can arise when dealing with non-linear problems.
- Consider the Burgers equation in the quasilinear form

$$u_t + uu_x = 0 \; . \tag{8}$$

A "natural" finite difference method can be obtained with a minor modification of the upwind method applied to the advection equation, assuming $v_j^n \geqslant 0$ for all j, n:

$$v_j^{n+1} = v_j^n - \frac{k}{h} v_j^n \left(v_j^n - v_{j-1}^n \right) . \tag{9}$$

This method converges on smooth solutions.

Exercise Set #5 - Exercise 3

Exercise 3: Finite Differences for Burger's Equation

- Compute the numerical solution obtained by (9), driven by the initial condition

$$u(x,0) = \begin{cases} 1 & x < 0 \\ 0 & x \geqslant 0 \end{cases} . \tag{10}$$

  Implement the method in Matlab and solve (8) up to $T = 0.5$ in the interval $(-1, 1)$ with initial condition (10). In your computations use $k = 0.5h$, $h = 0.01$.

- Is the solution obtained a weak solution? Is it the entropy solution?
- Now use the following initial condition in your code

$$u\left(x,0\right) = \begin{cases} 1.2 & x < 0 \\ 0.4 & x \geqslant 0 \end{cases} . \tag{11}$$

- Compare the solution to the known entropy solution for Burgers, which can be constructed by considering the characteristics and shocks.

Exercise Set #5 - Exercise 4

Exercise 4: Unconditionally Stable Method

- Apply the generalization of the Lax-Friedrichs method

$$v_j^{n+1} = \frac{1}{2}\left(v_{j+1}^n + v_{j-1}^n\right) - \frac{k}{2h}\left(f\left(v_{j+1}^n\right) - f\left(v_{j-1}^n\right)\right) \tag{12}$$

  to Burgers equation in conservation form,

$$u_t + \left(\frac{1}{2}\,u^2\right)_x = 0\ , \tag{13}$$

  with the initial conditions (10) and (11).

- Does the numerical solutions converge to a weak solution?
- Is this the entropy solution?
- Show that the generalization of the Lax-Friedrichs method to nonlinear conservation laws can be written in conservative form.

Exercise Set #5 - Exercise 4 - Hints

Exercise 4: Unconditionally Stable Method

- Does the numerical solutions converge to a weak solution? Yes, Why?
- Is this the entropy solution? Is the scheme montone? Prove it or disprove it
- Show that the generalization of the Lax-Friedrichs method to nonlinear conservation laws can be written in conservative form. Arrange terms so that the numerical flux looks like this

$$F\left(u, v\right) = \frac{1}{2}\left(f\left(v\right) + f\left(u\right)\right) - \frac{h}{2k}\left(v - u\right) \ . \tag{14}$$

Exercise Set #5 - Exercise 5

Exercise 5: Unconditionally Stable Method

- Solve the Burgers equation with the Engquist-Osher flux and the initial conditions (10) and (11).
- Does the solution converge to the entropy solution?
- How does the solution compare to that obtained with the Lax-Friedrichs scheme?

Exercise Set #5 - Exercise 5 - Hints

Exercise 5: Unconditionally Stable Method

- Q1. Implementation
- Q2. What do you think?
- Q3.
  - ✓ Both the Engquist-Osher flux and Lax-Friedrich flux can be written as

$$F(u,v) = \frac{1}{2}(f(u)+f(v)) - \frac{1}{2}Q(u,v)(v-u), \tag{15}$$

  - ✓ where $Q(u,v)$ is a measure of the artificial viscosity/dissipation introduced by the scheme.
  - ✓ For these two schemes, we have

$$Q^{LF}(u,v) = \frac{h}{k}, \quad Q^{EO}(u,v) = \frac{\int_u^v |f'(\xi)|d\xi}{v-u}. \tag{16}$$

  - ✓ How do $Q^{EO}(u,v)$ and $Q^{LF}(u,v)$ compare?