Exercise Class #4
Numerical Methods for Conservation Laws

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Exercise Class #4

Today's Topics: Exercise Set #4

- Numerical Methods for CL: The Finite Difference Method.
- Leapfrog Method
- Lax-Friedrichs Method
- Analysis
  - ✓ Consistency
  - ✓ Stability
  - ✓ Convergence

Exercise Set #4 - Exercise 1

Exercise 1: Method of Characteristics

- Define the following concepts:
  (a) Consistency;
  (b) Stability; and,
  (c) Convergence.
- Discuss the importance of these concepts, how they relate to each other, and how in general they can be established.

Exercise Set #3 - Exercise 1: Hints

Exercise 2: Finite Differences Method

- Consistency. A method is consistent if its local truncation error T_k satisfies

$$\tau_k(x, t) = O\left(k^p\right) + O\left(h^q\right) \qquad \text{where} \qquad p, q > 0\ . \tag{1}$$

  Essentially, consistency tells us that we are approximating the solution of the correct PDE.

- Stability. A method $v^{n+1} = \mathcal{H}_k v^n$ is stable if for each $T > 0$ there exist constants C and k_0, such that

$$\|\mathcal{H}_k^n\| \leqslant C \qquad\qquad 0 \leqslant nk \leqslant T\ , \quad 0 < k < k_0\ . \tag{2}$$

- Convergence. A method is convergent if the error E_k satisfies

$$\lim_{k\to 0} \max_{0 \leqslant kn \leqslant T} \|E_k\left(\cdot, kn\right)\| = 0\ . \tag{3}$$

Exercise Set #3 - Exercise 2

Exercise 2: Leapfrog Method

- The forward time, center space finite difference approximation to the advection equation $u_t + au_x = 0$ leads to a method which is *unstable*.
- So let us try something different.
- Leapfrog method. By approximating the time derivative by a centered difference, instead of the forward Euler discretization, we get the

$$v_j^{n+1} = v_j^{n-1} - \frac{ak}{h}\left(v_{j+1}^n - v_{j-1}^n\right) . \tag{4}$$

(i) Draw the stencil of (4).

(ii) Show that (4) is second order accurate in both space and time.

(iii) What is an obvious disadvantage of the Leapfrog method compared to the Lax-Friedrichs or Lax-Wendroff methods?

Exercise Set #4 - Exercise 2: Hints

Exercise 2: Leapfrog Method

(i) You may draw it by yourself.

(ii) The Leapfrofg Scheme is second order accurate in both space and time.

- ✓ Let u be a smooth solution of $u_t + au_x = 0$.
- ✓ The local truncation error for the Leapfrog scheme is given by

$$\begin{aligned} k\tau_k(x,t) =& u(x,t+k) - u(x,t-k) \\ &+ \frac{ak}{h}\left(u(x+h,t) - u(x-h,t)\right) \end{aligned} \tag{5}$$

- ✓ Next, we expand all the terms on the right hand side of (5) about (x,t).
- ✓ For example, we have $u(x,t+k)$ and $u(x,t-k)$ given by

$$u(x,t+k) = u + u_t k + \frac{1}{2}u_{tt}k^2 + O\left(k^3\right) \tag{6}$$

and

$$u(x,t-k) = u - u_t k + \frac{1}{2}u_{tt}k^2 + O\left(k^3\right) \ , \tag{7}$$

Exercise Set #4 - Exercise 2: Hints

Exercise 2: Leapfrog Method

(ii) (Continuation)

- ✓ You should obtain

$$\tau_k(x,t) = 2\left(u_t + a u_x\right) + O\left(k^2\right) + O\left(h^2\right) \ . \tag{8}$$

- ✓ Recall that u satisfies $u_t + a u_x = 0$.

(iii) Properties of the Le

- ✓ What do we need to store to compute v^{n+1} in the Leapfrog scheme?
- ✓ Initial Values?

Exercise Set #4 - Exercise 3

Exercise 3: Stability of the Lax-Friedrichs Method

- As a possible numerical method for the linear transport equation $u_t + au_x = 0$, consider the Lax-Friedrichs method

$$v_j^{n+1} = \frac{1}{2}\left(v_{j+1}^n + v_{j-1}^n\right) - \frac{ak}{2h}\left(v_{j+1}^n - v_{j-1}^n\right) \tag{9}$$

- Show that this method is stable in the l^∞ norm, provided that k and h satisfy the *CFL condition*

$$\frac{|a|k}{h} \leqslant 1. \tag{10}$$

Exercise Set #4 - Exercise 3 - Hints

Exercise 3: Stability of the Lax-Friedrichs Method

- To show that the Lax-Friedrichs (LF) scheme is stable provided

$$\frac{|a|k}{h} \leqslant 1 \,, \tag{11}$$

  we show that $\|\mathcal{H}^n\|_\infty$ is bounded for all n, where $\mathcal{H}$ is the operator defined by

$$\mathcal{H}v_j = \frac{1}{2}\left(v_{j+1} + v_{j-1}\right) - \frac{ak}{2h}\left(v_{j+1} - v_{j-1}\right) \,. \tag{12}$$

- That is, the LF scheme is given by $v^{n+1} = \mathcal{H}v^n$.
- To do so, we suppose v is some grid function, and calculate the norm of $\mathcal{H}v$.
- Prove

$$\|\mathcal{H}^n\|_\infty \leqslant \|\mathcal{H}\|_\infty^n \leqslant 1 \qquad \forall n \in \mathbb{N} \,. \tag{13}$$

Exercise Set #4 - Exercise 4

Exercise 4: Unconditionally Stable Method

- We have seen in the previous exercise that the Lax-Friedrichs method is stable provided k and h satisfy a CFL condition.
- A method which is stable for any k and h is said to be unconditionally stable.
- For $a > 0$, prove that the following backward-time backward-space method

$$v_j^{n+1} = v_j^n - \frac{ak}{h}\left(v_j^{n+1} - v_{j-1}^{n+1}\right)$$

  is unconditionally stable in the l^∞ norm.
- Assume periodic boundaries.

Exercise Set #4 - Exercise 4

Exercise 4: Unconditionally Stable Method

- Consider the scheme

$$v_j^{n+1} = v_j^n - \lambda\left(v_j^{n+1} - v_{j-1}^{n+1}\right), \quad \lambda = \frac{ak}{h}. \tag{14}$$

- Re-write the scheme (assuming periodic boundaries) as

$$(1+\lambda)\, v_j^{n+1} - \lambda v_{j-1}^{n+1} = v_j^n \quad \Longrightarrow \quad Av^{n+1} = v_n, \tag{15}$$

- Find A and write the scheme in the form $v^{n+1} = \mathcal{H} v^n$, where $\mathcal{H} = A^{-1}$,
- Prove that A is non-singular by usin Gershgorin's theorem.
- Use the following result.

$$\|A^{-1}\|_\infty \leqslant \frac{1}{\alpha}, \quad \alpha = \min_k\left(|A_{kk}| - \sum_{j\neq k}|A_{kj}|\right). \tag{16}$$

Exercise Set #4 - Exercise 5

Exercise 5: Unconditionally Stable Method

- Consider the scalar advection equation $u_t + au_x = 0$ with $a = 1$.
- Let u be the solution of $u_t + au_x = 0$ in $(-1, 1)$ that satisfies initial condition

$$u(x, 0) = u_0(x), \tag{17}$$

where

$$u_0(x) = \begin{cases} 1 & x < 0 \\ 0 & x > 0 \end{cases}, \tag{18}$$

and boundary conditions

$$u(-1, t) = 1 \qquad u(1, t) = 0 \tag{19}$$

- We already know that the exact solution for $0 < t < 1$ is given by $u_0(x - at)$,
- How well do numerical methods approximate such a problem?

Exercise Set #4 - Exercise 5

Exercise 5: Unconditionally Stable Method

- In the following consider the schemes

$$
\begin{aligned}
\text{Upwind:} \quad & v_j^{n+1} = v_j^n - \frac{ak}{h}(v_j^n - v_{j-1}^n) \\
\text{Lax-Friedrichs:} \quad & v_j^{n+1} = \frac{1}{2}(v_{j+1}^n + v_{j-1}^n) - \frac{ak}{2h}(v_{j+1}^n - v_{j-1}^n) \\
\text{Lax-Wendroff:} \quad & v_j^{n+1} = v_j^n - \frac{ak}{2h}(v_{j+1}^n - v_{j-1}^n) \\
& \quad + \frac{(ak)^2}{2h^2}(v_{j+1}^n - 2v_j^n + v_{j-1}^n) \\
\text{Beam-Warming:} \quad & v_j^{n+1} = v_j^n - \frac{ak}{2h}(3v_j^n - 4v_{j-1}^n + v_{j-2}^n) \\
& \quad + \frac{(ak)^2}{2h^2}(v_j^n - 2v_{j-1}^n + v_{j-2}^n) \, .
\end{aligned}
$$