Exercise Class #3
Numerical Methods for Conservation Laws

Professor: Martin Licht
Assistant: Fernando Henríquez

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Exercise Class #3

Today's Topics: Exercise Set #3

- Methods of Characteristics (Continuation)
- Weak Solutions
- Entropy Solutions
- Rankine-Hugoniot Condition Questions?

Exercise Sheet #3 - Exercise 1

Exercise 1: Method of Characteristics

- Consider the conservation law

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0, \quad u(x,0) = u_0(x), \tag{1}$$

  and $f(u) = u^2/2$.

- Consider the following initial conditions

$$u_0(x) = \begin{cases} 1, & \text{for} \quad x < -1, \\ 0, & \text{for} \quad -1 < x < 1, \\ -1, & \text{for} \quad x > 1 \end{cases} \tag{2}$$

  and

$$u_0(x) = \begin{cases} -1, & \text{for} \quad x < -1, \\ 0, & \text{for} \quad -1 < x < 1, \\ 1, & \text{for} \quad x > 1. \end{cases} \tag{3}$$

- Draw the profile of $u_0(x)$ and sketch the characteristics of the entropy solution $u(x,t)$ in the $x - t$ plane.

- Determine $u(x,t)$ for all $t > 0$.

Exercise Sheet #3 - Exercise 1

Theorem 4.12. *Let $q \in C^2(\mathbb{R})$ be strictly convex and $q'' \geq h > 0$ or strictly concave and $q'' \leq -h < 0$.*

a) *If $q'' > h$ and $u_- > u_+$ or $q'' < h$ and $u_- < u_+$, the unique entropy solution is given by the shock wave*

$$u(x,t) = \begin{cases} u_+ & x > \sigma(u_-, u_+)\,t \\ u_- & x < \sigma(u_-, u_+)\,t \end{cases} \tag{4.65}$$

where

$$\sigma(u_-, u_+) = \frac{q(u_+) - q(u_-)}{u_+ - u_-}.$$

b) *If $q'' > h$ and $u_- < u_+$ or $q'' < -h$ and $u_- > u_+$, the unique entropy solution is given by the rarefaction wave*

$$u(x,t) = \begin{cases} u_- & x < q'(u_-)\,t \\ r\left(\frac{x}{t}\right) & q'(u_-)\,t < x < q'(u_+)\,t \\ u_+ & x > q'(u_+)\,t \end{cases}$$

where $r = (q')^{-1}$ is the inverse function of q'.

Exercise Sheet #3 - Exercise 1: Hints

Exercise 2: Method of Characteristics

- We start with the solution driven by the initial data

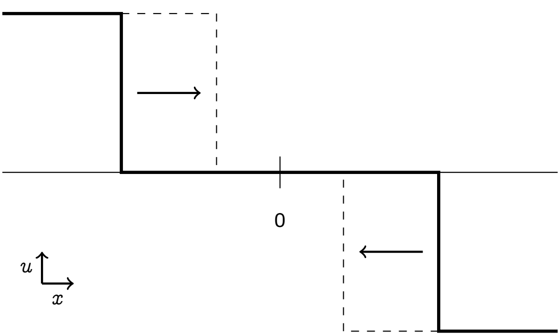
$$u_0(x) = \begin{cases} 1 & x < -1 \\ 0 & -1 < x < 1 \\ -1 & x > 1 \end{cases} . \tag{4}$$

- For Burger's equation, a shock forms if $u_l > u_r$.
- Both discontinuities in the initial function produce shocks.
- Shock's Speed? $\Rightarrow$ Rankine-Hugoniot jump condition

$$s = \frac{f(u_l) - f(u_r)}{u_l - u_r} . \tag{5}$$

- The left shock moves to the right and the right shock moves to the left.
- From this we expect the two shocks to meet at $x = 0$ at time $t = 2$.

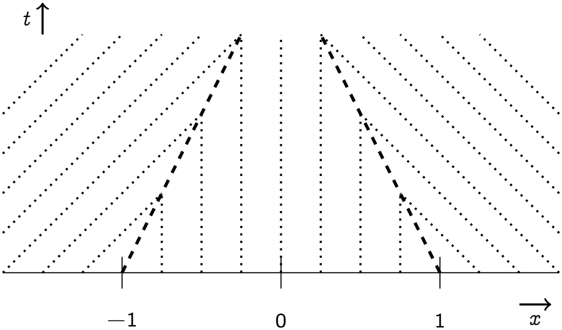
Exercise Sheet #3 - Exercise 1



(a)

Figure: Initial data u_0 given by (4), applied to the inviscid burgers equation. The two shocks move towards each other and merge at $x = 0$. At $x = 0$ they form a new stationary shock. (a) Initial data. (b) Characteristics.

Exercise Sheet #3 - Exercise 1



(a)

Figure: Initial data u_0 given by (4), applied to the inviscid burgers equation. The two shocks move towards each other and merge at $x = 0$. At $x = 0$ they form a new stationary shock. (a) Initial data. (b) Characteristics.

Exercise Sheet #3 - Exercise 2

Exercise 2: Traffic Flow Equation

- Consider the traffic flow equation

$$\frac{\partial q}{\partial t} + \frac{\partial}{\partial x} f(q) = 0, \tag{6}$$

  where $q(x, t)$ is the car density

- $U(q)$ is the traffic speed as a function of the car density and $f(q) = qU(q)$.
- A simple model for traffic flow is obtained by taking

$$U(q) = u_m \left(1 - \frac{q}{q_m}\right), \tag{7}$$

  with $u_m > 0$ and q_m being the maximum speed and maximum car density, respectively.

Exercise Sheet #3 - Exercise 2

Exercise 2: Traffic Flow Equation

- If the car density is maximal, we say that the traffic is "bumper-to-bumper".
- At zero density (empty road), the traffic speed is u_m.
- As q approaches q_m, the speed decreases to zero.
- The model then reads

$$\frac{\partial q}{\partial t} + \frac{\partial}{\partial x} u_m \left(q - \frac{q^2}{q_m} \right) = 0. \tag{8}$$

We equip (8) with the initial condition

$$q(x, 0) = \begin{cases} q_l, & \text{for} \quad x < 0, \\ q_r, & \text{for} \quad x > 0. \end{cases} \tag{9}$$

Exercise Sheet #3 - Exercise 2

Exercise 2: Traffic Flow Equation

(a) **The green light problem.**

- ✓ To represent this situation, we set $q_l = q_m$ and $q_r = 0$ in (9).
- ✓ Draw the profile of $q_0(x)$ and sketch the characteristics of the entropy solution $q(x,t)$ in the $x - t$ plane, and determine $u(x,t)$ for all $t > 0$.

(b) **Traffic jam ahead.**

- ✓ Consider $q_l = \frac{1}{8} q_m$ and $q_r = q_m$.
- ✓ The cars on the left move with speed $U = \frac{7}{8} u_m$, so that we expect congestion.

(c) **Entropy Solutions.** Show that for the traffic flow equation (8), the condition $q_l < q_r$ is required for a shock to be admissible. Verify:

(i) The entropy condition $f'(q_l) > s > f'(q_r)$.

(ii) There exists an entropy function $\eta(q)$ and a corresponding entropy flux $\psi(q)$ such that

$$s(\eta(q_r) - \eta(q_l)) \geqslant \psi(q_r) - \psi(q_l) \tag{10}$$

holds if and only if $q_l < q_r$.

Exercise Sheet #3 - Exercise 2: Hints

Exercise 2: The Green Light Problem.

- Observe that (8) can be written as $\frac{\partial q}{\partial t} + u_m \left(1 - 2\frac{q}{q_m}\right) \frac{\partial q}{\partial x} = 0$.
- Characteristics

$$\frac{dt(s)}{ds} = 1, \qquad \frac{dx(s)}{ds} = u_m \left(1 - 2\frac{q}{q_m}\right), \qquad \frac{dz(s)}{ds} = 0 \tag{11}$$

Considering the starting point $(0, \xi, q_0(\xi))$ of the characteristic curve, we get

$$t(s) = s, \quad z(s) = q_0(\xi) \tag{12}$$

- The solution to the problem is constant along the characteristic curve

$$x(t) = -u_m t + \xi, \text{ for } \xi \leqslant 0, \qquad x(t) = u_m t + \xi, \text{ for } \xi > 0. \tag{13}$$

- Rarefaction Wave

$$q(x,t) = \begin{cases} q_m & x < -u_m t \\ \frac{q_m}{2}\left(1 - \frac{x}{u_m t}\right) & -u_m t < x < u_m t \\ 0 & x \geqslant u_m t \end{cases} \tag{14}$$

Exercise Sheet #3 - Exercise 2: Hints

Exercise 2: Entropy Solutions.

- For the sake of simplicity, we assume $q_m = 1$.
- The flux function f is given by $f(q) = u_m \left(q - q^2\right)$.
- We choose $\eta(q) = q^2$ as an entropy function. By

$$\psi'(q) = \eta'(q) f'(q)$$

  we deduce the corresponding entropy flux to be

$$\psi(q) = u_m \left(q^2 - \frac{4}{3} q^3\right) \, .$$

- Insert ψ and η into

$$s\left(\eta\left(q_r\right) - \eta\left(q_l\right)\right) > \psi(q_r) - \psi(q_l) \tag{15}$$

Exercise Sheet #3 - Exercise 3

Exercise 3: Liu's Entropy Condition

- Consider the conservation law

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0 \tag{16}$$

  where $f(u) = e^u$.

- We consider the Riemann problem

$$u_0(x) = \begin{cases} u_l, & \text{for} \quad x < 0, \\ u_r, & \text{for} \quad x > 0 \end{cases} \tag{17}$$

  ✓ Show that when $u(x,t)$ is a weak solution and $\lambda > 0$, then $u(\lambda x, \lambda t)$ is a weak solution too.

  ✓ Suppose that the solution has a shock wave with values u_l and u_r to the left and to the right, respectively. Which values u_l and u_r are admissible on the basis of Liu's entropy condition?

Exercise Sheet #3 - Exercise 3

Exercise 3: Liu's Entropy Condition

- Liu's entropy condition means that a shock is admissible if the states u_l and u_r satisfy the condition

$$\frac{f(u^\star) - f(u_l)}{u^\star - u_l} \geqslant \frac{f(u_r) - f(u^\star)}{u_r - u^\star} \tag{18}$$

  where $u^\star \in (\min\{u_l, u_r\}, \max\{u_l, u_r\})$.

- This condition reflects that the characteristics must run into the shock.
- Requiring that the jump behind travels faster than the jump ahead, The flux $f(u) = e^u$ is strictly convex.
- Therefore, to have an admissible shock one requires $u_l < u_r$.