Exercise Class #2
Numerical Methods for Conservation Laws

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Friday 6th of October, 2023

Exercise Class #2

Today's Topics: Exercise Set #2

- Methods of Characteristics (Continuation)
- Weak Solutions
- Entropy Solutions
- Rankine-Hugoniot Condition
- Questions?

Exercise Sheet #2 - Exercise 1

Exercise 1: Method of Characteristics

- Suppose that the flux $f(u, x, t)$ is differentiable in all variables.
- Find curves along which the conservation law

$$\frac{\partial u(x,t)}{\partial t} - x\frac{\partial f(u(x,t),x,t)}{\partial x} = 0$$

can be written as a collection of ODEs.

Exercise Sheet #2 - Exercise 2

Exercise 2: Method of Characteristics

(i) Consider the conservation law

$$\frac{\partial u}{\partial t} - x\frac{\partial u}{\partial x} = 0 \tag{1}$$

with initial value

$$u(x, 0) = x. \tag{2}$$

Sketch the characteristics up to time $t = 1$. Describe the graph of the function $u(\cdot, t)$ as t increases.

(ii) Consider the conservation law

$$\frac{\partial u}{\partial t} + x\frac{\partial u}{\partial x} = 0 \tag{3}$$

with initial value

$$u(x, 0) = x. \tag{4}$$

Draw the characteristics and describe the graph of the function $u(\cdot, t)$ as t increases.

Exercise Sheet #2 - Exercise 2: Hints

Exercise 2: Method of Characteristics

- Set $F(t, x, z) = u(x, t) - z$, hence the solution of the conservation law

$$\frac{\partial u}{\partial t} - x \frac{\partial u}{\partial x} = 0 \tag{5}$$

- Observe that 5 can be written as $\begin{pmatrix} \frac{\partial u}{\partial t} \\ \frac{\partial u}{\partial x} \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -x \\ 0 \end{pmatrix} = 0.$

- We get the following set of ODEs describing the characteristic curve

$$\frac{dt(s)}{ds} = 1, \quad \frac{dx(s)}{ds} = -x, \quad \text{and} \quad \frac{dz(s)}{ds} = 0. \tag{6}$$

- Solution

$$t(s) = s, \quad x(s) = \xi \exp(-s), \quad \text{and} \quad z(s) = u_0(\xi). \tag{7}$$

From $F(t, x, z) = 0$, we get that

$$u(x, t) = u_0(\xi) = u_0(x \exp(t)) = x \exp(t). \tag{8}$$

Exercise Sheet #2 - Exercise 3: Hints

Exercise 3: Weak Solutions

- Show that a weak solution to the linear transport equation

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0,$$

with $a \in \mathbb{R}$ and initial data

$$u(x,0) = \begin{cases} 1, & \text{for} \quad x < 0, \\ 0, & \text{for} \quad x > 0, \end{cases} \tag{9}$$

is given by

$$u(x,t) = \begin{cases} 1, & \text{for} \quad x < at, \\ 0, & \text{for} \quad x > at, \end{cases} \tag{10}$$

Exercise Sheet #2 - Exercise 3: Hints

Exercise 3: Weak Solutions

- We need to prove that for all test functions $\phi \in C^1(\mathbb{R} \times [0, \infty))$ with compact support, it holds

$$\int_0^\infty \int_{-\infty}^\infty \left(\frac{\partial \phi}{\partial t} u + \frac{\partial \phi}{\partial x} f(u) \right) \mathrm{d}x \mathrm{d}t = - \int_{-\infty}^\infty \phi(x, 0) u(x, 0) \mathrm{d}x, \qquad (11)$$

  where $f(u) = au$ in the case of this exercise.

- Assume without loss of generality that $a > 0$.

- Then we calculate

$$\begin{aligned} \int_0^\infty \int_{-\infty}^\infty \frac{\partial \phi}{\partial t} u \, \mathrm{d}x \mathrm{d}t &= \int_{-\infty}^0 \int_0^\infty \frac{\partial \phi}{\partial t} \mathrm{d}t \mathrm{d}x + \int_0^\infty \int_{x/a}^\infty \frac{\partial \phi}{\partial t} \mathrm{d}t \mathrm{d}x \\ &= - \int_{-\infty}^0 \phi(x, 0) \mathrm{d}x - \int_0^\infty \phi\left(x, \frac{x}{a}\right) \mathrm{d}x, \end{aligned} \qquad (12)$$

- You may continue by yourself...

Exercise Sheet #2 - Exercise 4

Exercise 4: Method of Characteristics

- Consider the initial value problem

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0, \quad u(x,0) = u_0(x), \tag{13}$$

with $f(u) = \frac{u^2}{2}$, and

$$u_0(x) = \begin{cases} 2, & 0 < x < 1, \\ 0, & \text{otherwise}, \end{cases} \tag{14}$$

- Here a rarefaction wave arises at one discontinuity and a shock at the other.
- The goal of this exercise is to determine the exact solution for all $t > 0$.
- In this setup, the rarefaction wave catches up with the shock at some time $T_c > 0$.

Exercise Sheet #2 - Exercise 4

Exercise 4: Shocks and Rarefactions Waves

Consider the initial value problem

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0, \quad u(x,0) = u_0(x), \tag{15}$$

with $f(u) = \frac{u^2}{2}$, and

$$u_0(x) = \begin{cases} 2, & 0 < x < 1, \\ 0, & \text{otherwise}, \end{cases} \tag{16}$$

(i) Draw the profile of $u_0(x)$ and sketch the characteristics in the strip $0 < t < T_c$ of the $x - t$ plane.

(ii) Determine the exact solution for $0 < t < T_c$.

(iii) Let $x_s(t)$ be shock's location at $t > T_c$. By using the Rankine-Hugoniot jump condition construct an ODE to determine $x_s(t)$ for all $t > T_c$. In the sketch you drew in (i), extend the characteristic lines to $t > T_c$.

Exercise Sheet #2 - Exercise 4

Theorem 4.12. *Let $q \in C^2(\mathbb{R})$ be strictly convex and $q'' \geq h > 0$ or strictly concave and $q'' \leq -h < 0$.*

a) *If $q'' > h$ and $u_- > u_+$ or $q'' < h$ and $u_- < u_+$, the unique entropy solution is given by the shock wave*

$$u(x,t) = \begin{cases} u_+ & x > \sigma(u_-, u_+)\, t \\ u_- & x < \sigma(u_-, u_+)\, t \end{cases} \tag{4.65}$$

where

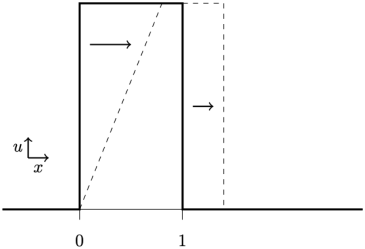
$$\sigma(u_-, u_+) = \frac{q(u_+) - q(u_-)}{u_+ - u_-}.$$

b) *If $q'' > h$ and $u_- < u_+$ or $q'' < -h$ and $u_- > u_+$, the unique entropy solution is given by the rarefaction wave*

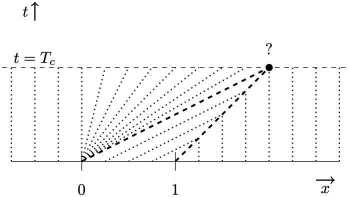
$$u(x,t) = \begin{cases} u_- & x < q'(u_-)\, t \\ r\left(\frac{x}{t}\right) & q'(u_-)\, t < x < q'(u_+)\, t \\ u_+ & x > q'(u_+)\, t \end{cases}$$

where $r = (q')^{-1}$ is the inverse function of q'.

Exercise Sheet #2 - Exercise 4



(a) Initial Condition



(b) Characteristics up until the rarefaction wave catches the shock.

Exercise Sheet #2 - Exercise 4

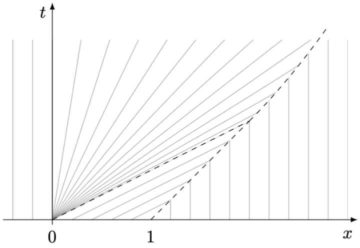


Figure: The characteristic curves (gray lines) of u in the $x - t$ plane.