Exercise Class #1
Numerical Methods for Conservation Laws

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Friday 28th of September, 2023

Exercise Class #1

Today's Topics: Exercise Sheet #1

- Methods of Characteristics (Continuation)
- Weak Solutions
  - ✓ Definition
  - ✓ Why are weak solutions important?
  - ✓ Verify that a given function is a weak solution.
  - ✓ Are weak solutions unique?
- Questions?

Exercise Sheet #0 - Exercise 1

Exercise 1: Method of Characteristics

- Conservation Law:

$$u_t + f(u)_x = 0 \ , \quad u(x,0) = u_0(x)$$

- Inviscid Burgers equation: $f(u) = u^2/2$.
- **Q:** Draw the characteristics of the solution in the x-t plane with I.C.

$$u(x,0) = u_0(x) := \begin{cases} 0 & x < -1 \\ 1+x & -1 < x < 0 \\ 1-x & 0 < x < 1 \\ 0 & 1 < x \end{cases} .$$

- **Q:** Compute the exact solution in $0 < t < 1$ and draw its profile at $t = 1$.

Exercise Sheet #0 - Exercise 1: Hints

Exercise 1: Method of Characteristics

- Burger's Equation: $u_t + uu_x = 0$.
- Characteristics's ODEs

$$\frac{d\hat{x}}{dt} = \hat{u}, \quad \frac{d\hat{u}}{dt} = 0,$$

- Initial Conditions

$$\hat{x}(\xi, 0) = \xi, \quad \hat{u}(\xi, 0) = u_0(\xi). \tag{1}$$

- We get

$$\hat{x}(\xi, t) = u_0(\xi)t + \xi. \tag{2}$$

It follows that

$$\hat{x}(\xi, t) = \begin{cases} \xi & \xi < -1 \\ (1+\xi)\,t + \xi & -1 < \xi < 0 \\ (1-\xi)\,t + \xi & 0 < \xi < 1 \\ \xi & 1 < \xi \end{cases}. \tag{3}$$

Exercise Sheet #1 - Exercise 1: Hints

Burger's Equation

- With $f(u) = u^2/2$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0.$$

- Method of Characteristics: Find an ODE to solve the PDE.
- Observe that

$$\begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial t} \\ -1 \end{pmatrix} \cdot \begin{pmatrix} u \\ 1 \\ 0 \end{pmatrix} = 0$$

- Consider the solution surface $F(x, t, z) = 0$, with $F(x, t, z) = u(x, t) - z$.
- Observe that

Normal to $F(x, t, z) = 0$ is $\begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial t} \\ -1 \end{pmatrix} \Rightarrow \begin{pmatrix} u \\ 1 \\ 0 \end{pmatrix}$ is a tangent vector to $F(x, t, z) = 0$.

Exercise Sheet #1 - Exercise 1: Hints

Exercise 1: Method of Characteristics

- For any initial point $(\xi, 0, u(\xi, 0))^\top$, we move with speed vector $(u, 1, 0)^\top$ to generate a curve on the solution surface $F(x, t, z) = 0$
- The tangent vector to a curve parametrized by $(x(s), t(s), z(s))^\top$ is

$$\left(\frac{dx(s)}{ds}, \frac{dt(s)}{ds}, \frac{dz(s)}{ds}\right)^\top$$

- Then

$$\frac{dx(s)}{ds} = u, \quad \frac{dt(s)}{ds} = 1, \quad \text{and} \quad \frac{dz(s)}{ds} = 0,$$

- Solution to he ODEs with initial condition $(\xi, 0, u_0(\xi))^\top$

$$x(s) = u_0(\xi)s + \xi, \quad t(s) = s, \quad \text{and} \quad z(s) = u_0(\xi).$$

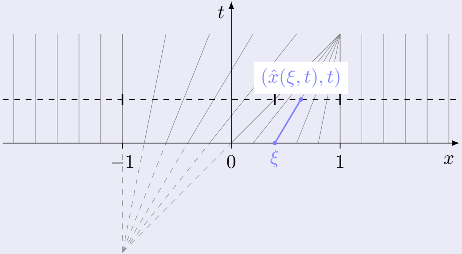
- Then $x(t) = u_0(\xi)t + \xi$.
- Compute $\xi(x, t)$ from $x(t, \xi)$ for $0 < t < 1$
- Then

$$z(t) = u_0(\xi) \Rightarrow u(x, t) = u_0(\xi(x, t)).$$

Exercise Sheet #1 - Exercise 1: Hints

Exercise 1: Method of Characteristics

- Characteristics Curves



- Characteristics intersect at $t = 1$!

$$\hat{x}(\xi, t) = \begin{cases} \xi & \xi < -1 \\ (1+\xi)\,t + \xi & -1 < \xi < 0 \\ (1-\xi)\,t + \xi & 0 < \xi < 1 \\ \xi & 1 < \xi \end{cases}.$$

Exercise Sheet #1 - Exercise 2

Exercise 2: Weak Solutions (Technical and Hard)

- Riemann problem: Consider the following IVP for the Burgers equation

$$u_t + \left(\frac{u^2}{2}\right)_x = 0 \quad u(x,0) = u_0(x) = \begin{cases} u_l & x < 0 \\ u_r & 0 < x \end{cases} . \tag{4}$$

- For $u_l < u_r$, and any $u_m \in (u_l, u_r)$, let

$$u(x,t) = \begin{cases} u_l & x < s_m t \\ u_m & s_m t < x < u_m t \\ x/t & u_m t < x < u_r t \\ u_r & u_r t < x \end{cases} , \tag{5}$$

  where $s_m = (u_m + u_l)/2$.

- **Q:** Show that u is a weak solution of (4), and draw its characteristics.
- **Q:** Can you give the expression of any other weak solution for this problem?

Exercise Sheet #1 - Exercise 2: Hints

Exercise 2: Hints

- We say that u is a weak solution of

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0,$$

if for any compactly supported C^1 function $\phi = \phi(x,t)$, there holds

$$\int_0^\infty \int_{-\infty}^\infty \left(\frac{\partial \phi}{\partial t} u + \frac{\partial \phi}{\partial x} f(u) \right) dx dt = -\int_{-\infty}^\infty \phi(x,0) u_0(x) dx.$$

- Set

$$B(\phi, u) = \int_0^\infty \int_{-\infty}^\infty \left(\frac{\partial \phi}{\partial t} u + \frac{\partial \phi}{\partial x} f(u) \right)$$

- We need to prove that

$$B(\phi, u) = -\int_{-\infty}^\infty \phi(x,0) u_0(x) dx.$$

Exercise Sheet #1 - Exercise 2: Hints

Exercise 2: Weak Solutions

- Consider the following decomposition of $u(x,t)$

$$u(x,t) = u_1(x,t) + u_2(x,t) - u_m,$$

where

$$u_1(x,t) = \begin{cases} u_l & x < s_m t \\ u_m, & x > s_m t \end{cases} \quad \text{and} \quad u_2(x,t) = \begin{cases} u_m & x < u_m t \\ x/t, & u_m t < x < u_r t \\ u_r, & u_r t < x \end{cases}$$

- Observe that

$$f(u(x,t)) = f(u_1(x,t)) + f(u_2(x,t)) - f(u_m).$$

and

$$B(\phi, u) = B(\phi, u_1) + B(\phi, u_2) - B(\phi, u_m).$$

Exercise Sheet #1 - Exercise 3

Exercise 3: Flux depends on x and $u(x,t)$

- Flux f depends on x and $u(x,t)$.
- Suppose u is the solution of the PDE

$$u_t + (a(x)u)_x = 0, \quad a(x) = x$$

  which satisfies the initial condition

$$u(x,0) = u_0(x).$$

- **Q:** Use the method of characteristics to compute the equation describing the characteristics in the x-t plane.
- **Q:** Is the solution u constant on the characteristic curves?

Exercise Sheet #1 - Exercise 3

Exercise 3: Flux depends on x and $u(x,t)$ - Hints

- With $f(x,u) = a(x)u$, $\frac{\partial u}{\partial t} + a(x)\frac{\partial u}{\partial x} = -\frac{da}{dx}u$
- **Method of Characteristics:** Find an ODE to solve the PDE.
- Observe that

$$\begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial t} \\ -1 \end{pmatrix} \cdot \begin{pmatrix} a(x) \\ 1 \\ -\frac{da}{dx} \end{pmatrix} = 0$$

- Consider the solution surface $F(x,t,z) = 0$, with $F(x,t,z) = u(x,t) - z$.
- Observe that

Normal to $F(x,t,z) = 0$ is $\begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial t} \\ -1 \end{pmatrix} \Rightarrow \begin{pmatrix} a(x) \\ 1 \\ -\frac{da}{dx}u \end{pmatrix}$ is tangent to $F(x,t,z) = 0$.

- For any initial point $(0, \xi, u(\xi, 0))^\top$, we follow the vector $(u, 1, 0)^\top$ to generate a curve on the solution surface $F(x,t,z) = 0$
- The tangent vector to a curve parametrized by $(x(s), t(s), z(s))^\top$.

Exercise Sheet #1 - Exercise 4

Exercise 4: Flux depends on x and $u(x,t)$

- Suppose $f(u) = au$, where $a > 0$ is a constant and $u_0(x)$ is an integrable function.
- Verify that $u(x) = u_0(x - at)$ satisfies

$$u_t + f(u)_x = 0 \ , \quad u(x,0) = u_0(x) \tag{6}$$

in integral form, i.e.

$$\int_{x_1}^{x_2} u(x,t_2)\,dx - \int_{x_1}^{x_2} u(x,t_1)\,dx = -\int_{t_1}^{t_2} f(u(x_2,t))\,dt + \int_{t_1}^{t_2} f(u(x_1,t))\,dt \ .$$