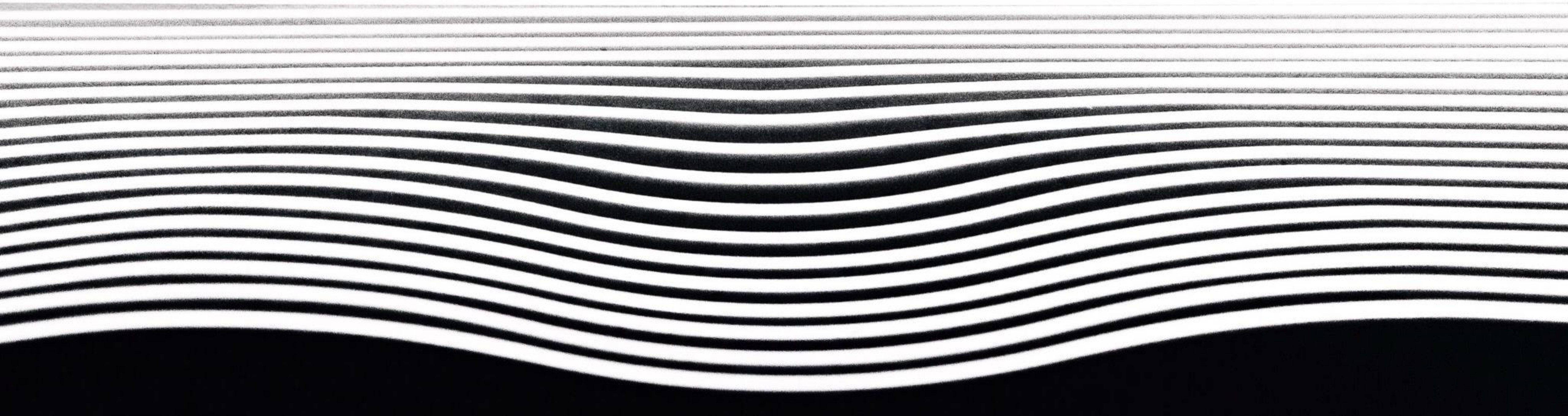
# Numerical methods for conservation laws
# 6: Finite Difference Schemes



Having recapitulated the essentials of the theory of conservation laws, we now address numerical methods.

We begin with finite difference schemes.

We introduce a few basic numerical schemes for the conservation law

$$\partial_t u + \partial_x f(u) = 0$$

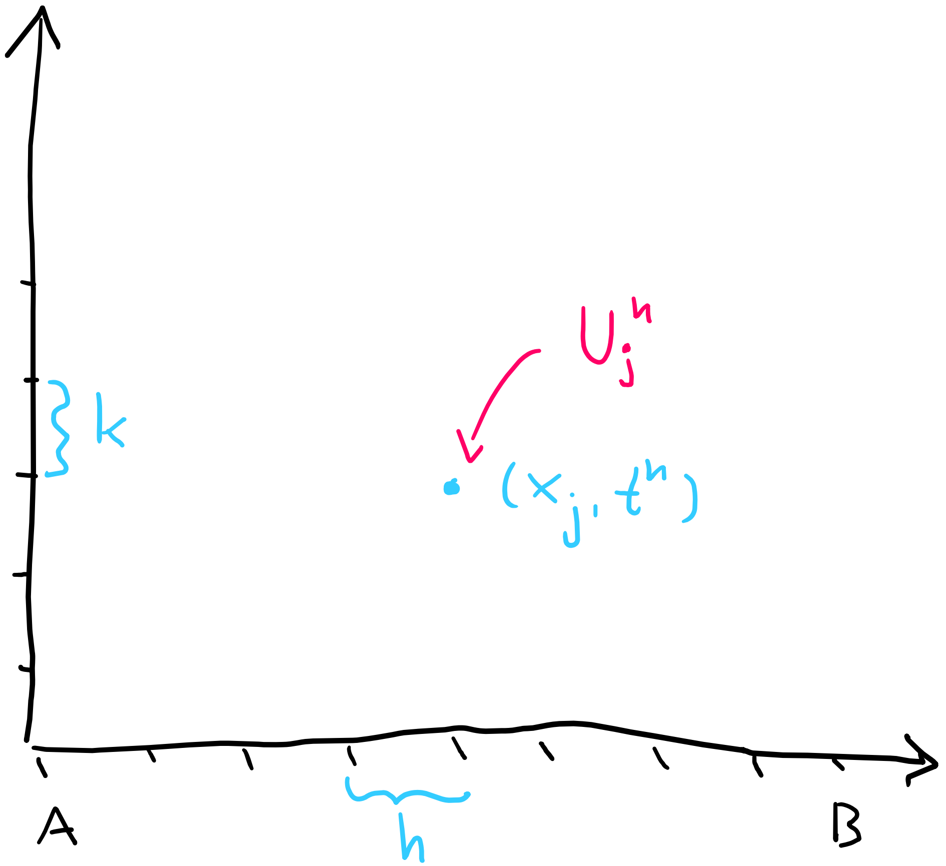
We work over a finite space-time domain $\Omega_x \times \Omega_t$,

$$\Omega_x = [A, B], \qquad \Omega_t = [0, T]$$

We introduce

- nodal points over $\Omega_x$ of equal distance $h$
- time steps over $\Omega_t$ over equal distance $k$.

We approximate the true solution by computing values $U_j^n$ on the space-time grid.



Specifically,

$$x_j = A + j \cdot h, \qquad j = 0, \dots, N \qquad h = \frac{B-A}{N}$$

$$t^n = n \cdot k, \qquad n = 0, \dots\dots, K \qquad k = T/K$$

This defines the discretization of space and time.

The function $u$ is discretized by associating the value $U_j^n$ to the space-time node $(x_j, t^n)$.

Given the data and boundary conditions, we want to compute values $U_j^n$ such that

$$U_j^n \approx u(x_j, t^n)$$

**How do we define $U_j^n$?**

Recall finite differences: we approximate the derivatives in the conservation law by difference quotients

Forward Difference:

$$\partial_x u(x_j, t^n) \approx \frac{U_{j+1}^n - U_j^n}{h} + \mathcal{O}(h) \qquad \partial_t u(x_j, t^n) \approx \frac{U_j^{n+1} - U_j^n}{k}$$

Backward Difference

$$\partial_x u(x_j, t^n) \approx \frac{U_j^n - U_{j-1}^n}{h} + \mathcal{O}(h) \qquad \partial_t u(x_j, t^n) \approx \frac{U_j^n - U_j^{n-1}}{k}$$

Central Difference

$$\partial_x u(x_j, t^n) \approx \frac{U_{j+1}^n - U_{j-1}^n}{2h} + \mathcal{O}(h^2) \qquad \left( \partial_t u(x_j, t^n) \approx \frac{U_j^{n+1} - U_j^{n-1}}{2k} \right)$$

Application: we replace the actual derivatives by finite differences. Then we isolate variables so that we can compute $U_j^n$ step by step.

**Example: Transport equation with constant speed**

$$\partial_t u + a\partial_x u = 0, \qquad u(x,0) = u_0(x)$$

1) We consider the scheme Forward-Time Backward-Space (FTBS):

$$\partial_t u + a\partial_x u = 0 \qquad \longleftrightarrow \qquad \frac{U_j^{n+1} - U_j^n}{k} + a\frac{U_j^n - U_{j-1}^n}{h} = 0$$

We isolate the variable $U_j^{n+1}$, leading to

$$U_j^{n+1} = U_j^n - \frac{ak}{h}(U_j^n - U_{j-1}^n)$$

Starting with $U_j^0 = u_0(x_j)$, we compute the solution timestep by timestep.

2) Alternatively, we use the scheme Forward-Time Forward-Space (FTFS):

$$\partial_t u + a\partial_x u = 0 \qquad \longleftrightarrow \qquad \frac{U_j^{n+1} - U_j^n}{k} + a\,\frac{U_{j+1}^n - U_j^n}{h} = 0$$

Isolating $U_j^{n+1}$ again, we get

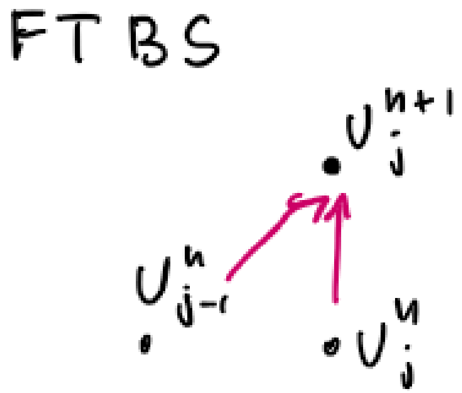
$$U_j^{n+1} = U_j^n - \frac{ak}{h}(U_{j+1}^n - U_j^n)$$

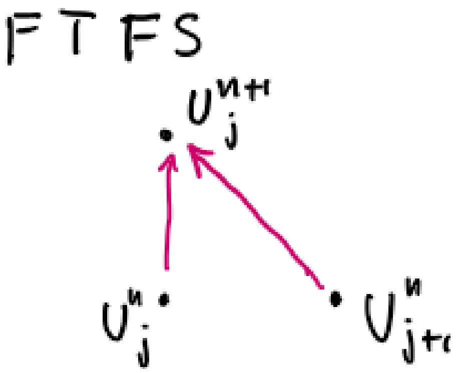
3) Similarly, we can define Forward-time centered space…

*Does the choice of method make a difference?*

Suppose we simulate the transport equation constant speed $a = 1$.

The finite-difference scheme should match the flow of information.
*Not only the direction matters, but also the speed.*




This is called **upwinding**: the first-order derivatives should be discretized such that the discrete flow of information matches the physical flux direction.

**Summary (Transport equation):**

We have a few finite-difference schemes with forward difference quotients in time.

FTBS $$U_j^{n+1} = U_j^n - c\frac{k}{h}\left(U_j^n - U_{j-1}^n\right)$$

FTFS $$U_j^{n+1} = U_j^n - c\frac{k}{h}\left(U_{j+1}^n - U_j^n\right)$$

FTCS $$U_j^{n+1} = U_j^n - \frac{ck}{2h}\left(U_{j+1}^n - U_{j-1}^n\right)$$

**We could also backward difference quotients in time (BTBS,BTFS,BTCS). That requires solving a linear system at each step.**

**Example: Burgers' equation**

Recall that Burgers' equation has two equivalent forms

$$\partial_t u + \partial_x(u^2) = 0 \quad \Longleftrightarrow \quad \partial_t u + 2u \cdot \partial_x u = 0$$

These inspire two different finite-difference schemes. For example, FTBS:

$$\frac{1}{k}\left(U_j^{n+1} - U_j^n\right) + \frac{(U_j^n)^2 - (U_{j-1}^n)^2}{h} = 0 \qquad U_j^{n+1} = U_j^n - \frac{k}{h}\left((U_j^n)^2 - (U_{j-1}^n)^2\right)$$

$$\frac{1}{k}\left(U_j^{n+1} - U_j^n\right) + \frac{2U_j^n}{h}\left(U_j^n - U_{j-1}^n\right) = 0 \qquad U_j^{n+1} = U_j^n - \frac{2k}{h} U_j^n \left(U_j^n - U_{j-1}^n\right)$$

Let $\Omega = [0,1]$ with periodic BC, $h = k = 1/100$ and $u_0 = 1 + \sin(2\pi x)$.

The first scheme works reasonably well, the second one does not. Why?

$$\sum_j U_j^{n+1} = \sum_j U_j^n - \frac{k}{h}\underbrace{\sum_j \left(U_j^n\right)^2 - \left(U_{j-1}^n\right)^2}_{=0}$$

$$\sum_j U_j^{n+1} = \sum_j U_j^n - \frac{2k}{h}\sum_j U_j^n\left(U_j^n - U_{j-1}^n\right)$$

$$= \sum_j U_j^n - \frac{2k}{h}\sum_j \left(U_j^n\right)^2 + \frac{2k}{h}\sum_j U_j^n U_{j-1}^n$$