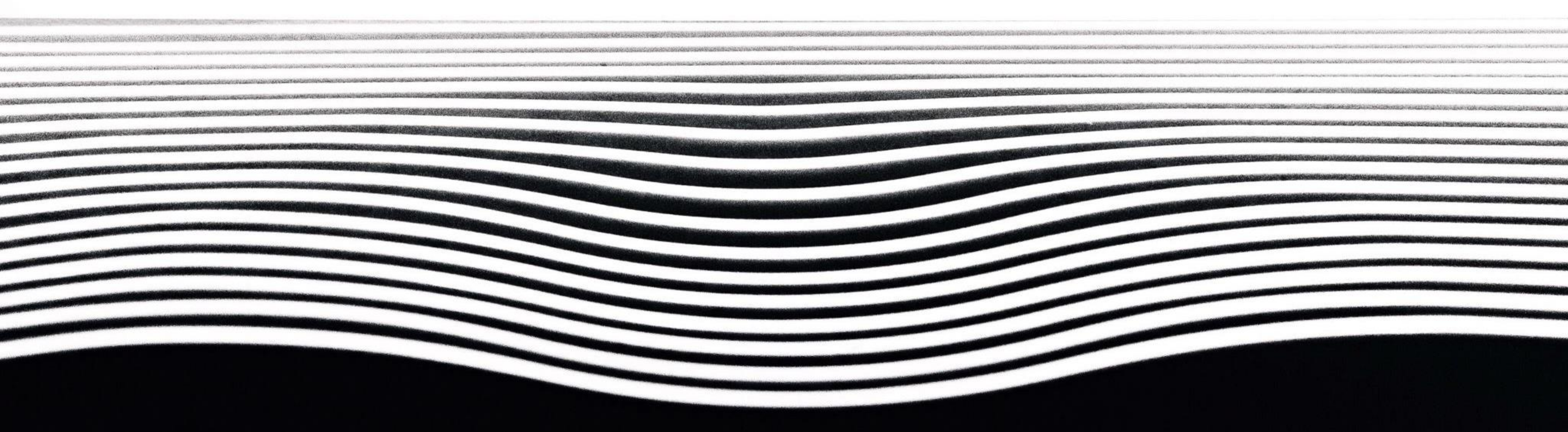
Numerical methods for conservation laws
5: Applications of entropy solutions



Agenda

- We were confronted with the problem that conservation laws admit multiple weak solutions. How to choose the physical weak solution?
- For a scalar conservation law in 1D, we discussed exclusion criteria (Lax entropy condition, Liu entropy condition)
- We defined **entropy solutions** as the unique physical weak solution, using entropy entropy-flux pairs.
- **Here:** We now revisit scalar conservation laws in 1D and apply the concept of entropy solution in that context.

Example 1:

The transport equation with constant speed

$$\partial_t u + a\partial_x u = 0, \qquad u(x,0) = u_0(x) = \begin{cases} u_l & x < 0 \\ u_r & x > 0 \end{cases}$$

has a weak solution $u(x,t) = u_0(x - at)$. We verify that this is an entropy solution.

Let (η, ψ) be a smooth entropy-entropy flux pair. Then

$$\psi'(u) = f'(u) \cdot \eta'(u) = a \cdot \eta'(u)$$

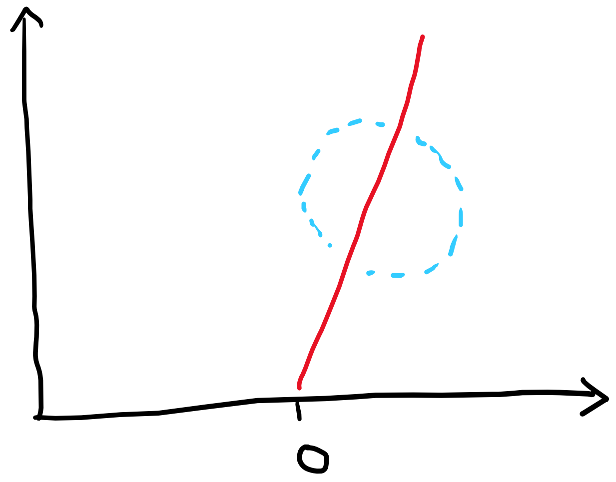
Hence

$$\psi(u) = a \cdot \eta(u) + C$$

For any non-negative test function φ.

1) Away from the discontinuity, u is a classical solution, and so $y(u)_t + \Psi(u)_x = 0$

2) Suppose φ is supported on a small domain over the discontinuity



We proceed similar as in the discussion of the RH condition...

$$\iint_{\omega} \eta(u)\, \varphi_t + a\, \eta(u)\, \varphi_x \quad dx\,dt$$

$$= \iint_{\omega_\ell} \eta(u_\ell)\, \varphi_t + a\, \eta(u_\ell)\, \varphi_x \quad dx\,dt \quad + \quad \iint_{\omega_r} \eta(u_r)\, \varphi_t + a\, \eta(u_r)\, \varphi_x \quad dx\,dt$$

$$= \oiint_{\partial\omega_\ell} \begin{pmatrix} a\, \eta(u_\ell) \\ \eta(u_\ell) \end{pmatrix} \frac{\begin{pmatrix} 1 \\ -a \end{pmatrix}}{\sqrt{1+a^2}}\, \varphi \quad ds \quad + \quad \oiint_{\partial\omega_r} \begin{pmatrix} a\, \eta(u_r) \\ \eta(u_r) \end{pmatrix} \frac{\begin{pmatrix} -1 \\ a \end{pmatrix}}{\sqrt{1+a^2}}\, \varphi \quad ds$$

$$= 0$$

This weak solution is an entropy solution! ☺

Example 2:

Conservation law with differentiable convex flux:

$$\partial_t u + f'(u)\,\partial_x u = 0, \qquad u(x,0) = u_0(x) = \begin{cases} u_l & x < 0 \\ u_r & x > 0 \end{cases} \qquad u_l > u_r$$

Since $f'(u)$ increases in u, the characteristics on the left or faster than the ones on the right, and we have a shock wave with speed

$$s = \frac{f(u_l) - f(u_r)}{u_l - u_r}$$

Let (η, ψ) be a smooth entropy–entropy flux pair. Then

$$\psi'(u) = f'(u) \cdot \eta'(u)$$

Let φ be a non-negative test function

1) Away from the shock, u is a classical solution

Example 2:

Burgers' equation with shock wave:

$$\partial_t u + 2u\,\partial_x u = 0, \qquad u(x,0) = u_0(x) = \begin{cases} u_l & x < 0 \\ u_r & x > 0 \end{cases} \qquad u_l > u_r$$

We verify that $u(x,t) = u_0(x - st)$ is an entropy solution, with shock speed

$$s = \frac{u_l^2 - u_r^2}{u_l - u_r}$$

Let (η, ψ) be a smooth entropy–entropy flux pair. Then

$$\psi'(u) = f'(u) \cdot \eta'(u) = 2u \cdot \eta'(u)$$

Let φ be a non-negative test function

1) Away from the shock, u is a classical solution. As before...

2) Suppose φ is supported on a small domain over the discontinuity.

$$\iint_{\omega} \eta(u)\,\varphi_t + \psi(u)\,\varphi_x \;dx\,dt = \cdots$$

$$= \oint_{\partial\omega_\ell} \begin{pmatrix} \psi(u_\ell) \\ \eta(u_\ell) \end{pmatrix} \frac{\begin{pmatrix} 1 \\ -s \end{pmatrix}}{\sqrt{1+s^2}} \varphi \; ds \quad + \quad \oint_{\partial\omega_r} \begin{pmatrix} \psi(u_r) \\ \eta(u_r) \end{pmatrix} \frac{\begin{pmatrix} -1 \\ s \end{pmatrix}}{\sqrt{1+s^2}} \varphi \; ds$$

$$= \int_C \Big[\psi(u_\ell) - \psi(u_r) - s\big(\eta(u_\ell) - \eta(u_r)\big) \Big] \frac{\varphi}{\sqrt{1+s^2}} \; ds$$

We need to show ≥ 0, that is,

$$\psi(u_\ell) - \psi(u_r) \geq s\big(\eta(u_\ell) - \eta(u_r)\big)$$

We define

$$S(u) = \frac{f(u) - f(u_r)}{u - u_r} \qquad E(u) = s(u)\left[y(u) - y(u_r)\right] - \left[\Psi(u) - \Psi(u_r)\right]$$

Note that $s(u_r) = f'(u_r)$. We want to show $E(u_\ell) \leq 0$.

Since $u_r < u_\ell$ and $E(u_r) = 0$,

we need to show $E'(u) \leq 0$ for $u \geq u_r$.

$$\begin{aligned} E'(u) &= s'(u)\big(y(u) - y(u_r)\big) + s(u)\, y'(u) - \Psi'(u) \\ &= s'(u)\big(y(u) - y(u_r)\big) + y'(u)\big(s(u) - f'(u)\big) \end{aligned}$$

By definition of $s(u)$ and the quotient rule:

$$s'(u) = \frac{(u-u_r)f'(u) - [f(u) - f(u_r)]}{(u-u_r)^2} = \frac{f'(u) - s(u)}{u - u_r}$$

Since f is convex, $f'(u) \geq s(u)$ for $u \geq u_r$.
Hence $s'(u) \geq 0$ for $u \geq u_r$

We substitute and find

$$\begin{aligned} E'(u) &= s'(u)\big(y(u) - y(u_r)\big) + y'(u)\big(s(u) - f'(u)\big) \\ &= s'(u)\big(y(u) - y(u_r)\big) - s'(u)\,y'(u)\,(u - u_r) \\ &= s'(u)\big(y(u) - y(u_r) - y'(u)(u - u_r)\big) \end{aligned}$$

$$E'(u) = S'(u)\Big(y(u) - y(u_r) + y'(u)(u_r - u) \Big)$$

We have

- $S'(u) \geq 0$ for $u \geqslant u_r$
- $y'(u)(u_r - u) \leq y(u_r) - y(u)$

It follows that $E'(u) \leq 0$ for $u \geqslant u_r$

Example 3:

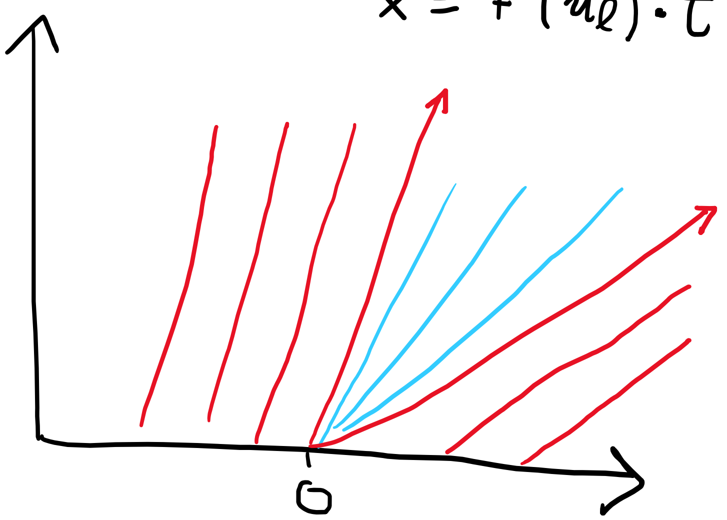
Conservation law with twice differentiable strictly convex flux:

$$\partial_t u + f'(u)\,\partial_x u = 0, \qquad u(x,0) = u_0(x) = \begin{cases} u_l & x < 0 \\ u_r & x > 0 \end{cases} \qquad u_r > u_l$$

Since $f'(u)$ increases in u, the characteristics on the right are than the ones on the left. We fill up the gap with characteristics that emerge from the origin (not from the shocks). Along those characteristics:

$$x = f'(u)t \quad \Leftrightarrow \quad \frac{x}{t} = f'(u) \quad \Leftrightarrow \quad (f')^{-1}\left(\frac{x}{t}\right) = u$$

Note that f' increases strictly monotonely from u_l to u_r.



$$u(x) = \begin{cases} u_\ell & x < f'(u_\ell)\cdot t \\ f'(u) = \frac{x}{t} & f'(u_\ell) < \frac{x}{t} < f'(u_r) \\ u_r & x > f'(u_r)\cdot t \end{cases}$$

1) Away from the two lines, u is a classical solution

2) For a test function φ supported on one of the lines, an integration by parts argument shows that

$$\int y(u)\varphi_t + \Psi(u)\varphi_x \, dx\,dt = 0$$

Theory is done! …. For now

Let’s do a

Summary

Summary

- Conservation laws: $\partial_t u + \partial_x f(u) = 0$
- The integral of u is conserved
- Several layers of complexity:
  - Scalar conservation laws (mostly understood)
  - Systems in 1D (mostly understood, tbc.)
  - Systems in higher dimensions still research area
- Important examples
  - Transport equation
  - Burgers' equation
  - Wave equation (system)
- The mathematical model is often the vanishing viscosity limit of a more complex physical model.

Summary

- Method of characteristics:
  - Easiest way to find the solution
- Problem: discontinuities arise in finite time when characteristics cross (shock wave) and the characteristics do not determine the solution everywhere
- Weak solutions
  - Broad enough notion of solutions.
  - Implies Rankine-Hugoniot condition.
- Problem: there is no unique weak solution.

Summary

- **Entropy conditions** (for scalar conservation laws in 1D)
  - Intention is to detect unphysical weak solutions
  - Lax entropy condition
  - Liu entropy condition
- Weak **entropy solutions** are the correct notion of solution
  - Definition via entropy-entropy flux pairs
  - L1 contraction property
  - Vanishing viscosity limits (if they exist) are entropy solutions
  - General existence results
  - Total variation
- Applications to scalar conservation laws in 1D
  - Strictly convex flux as generalization of Burgers' equation

What’s next?

- Finite Difference Schemes
- Finite Volume Schemes
- Systems of Conservation Laws
- ENO Schemes
- Discontinuous Galerkin schemes