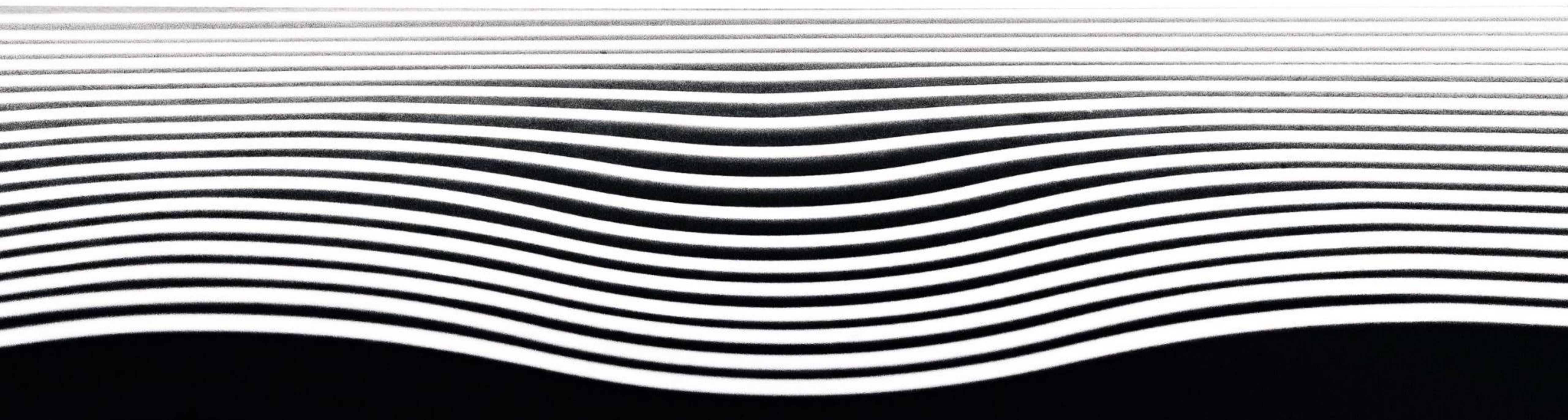
Numerical methods for conservation laws
4: Entropy conditions, Part 1



Recap

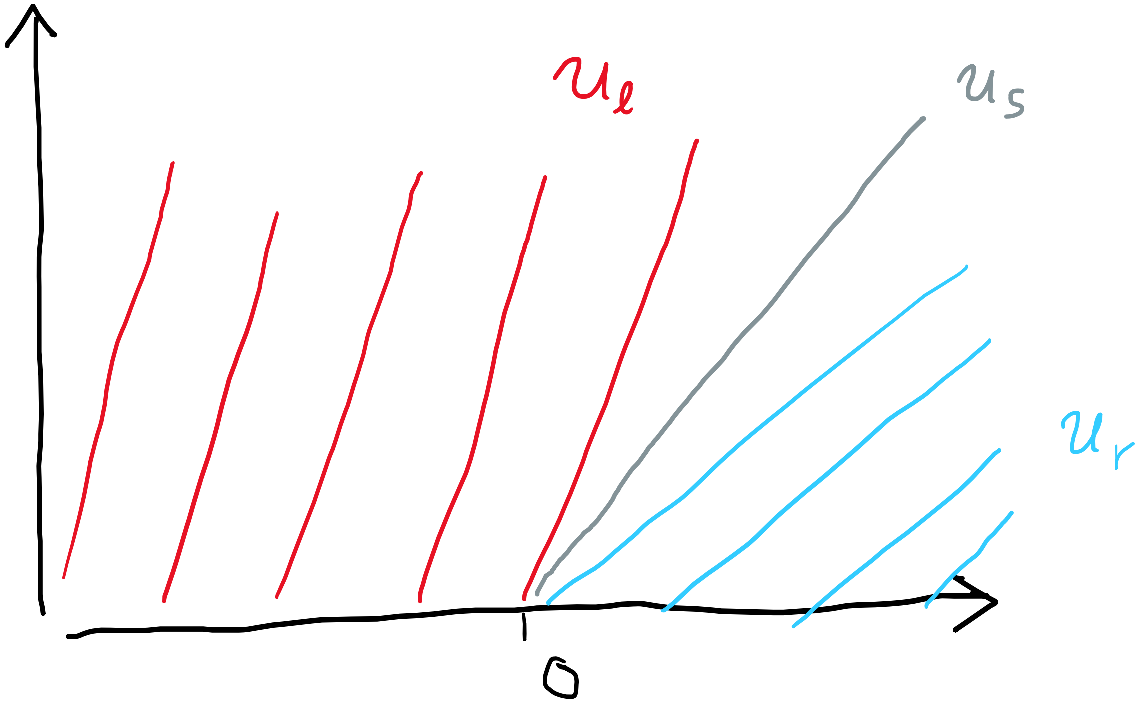
We have seen that *weak solutions* are natural for conservation laws.

However, there is no unique weak solution.

Consequently, we need criteria to single out the “*correct*” or “*physical*” weak solution.

So we have multiple weak solutions. How do we pick the one that's physical?

We want the characteristics to run into the shock curves, never to emerge from the shock curve.



Intuitively, no information is created at the shocks. Information can only vanish into the shock curves.

If the flux is differentiable, then the conservation can be written

$$\partial_t u + f'(u) \cdot \partial_x u = 0$$

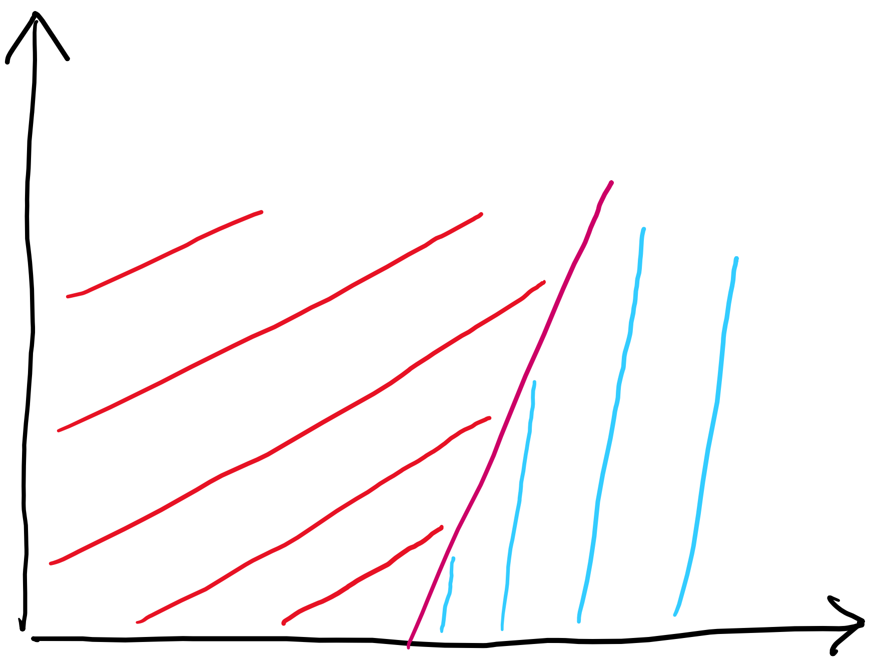
Along a shock curve with speed s, the characteristics shall run into the shock. Hence

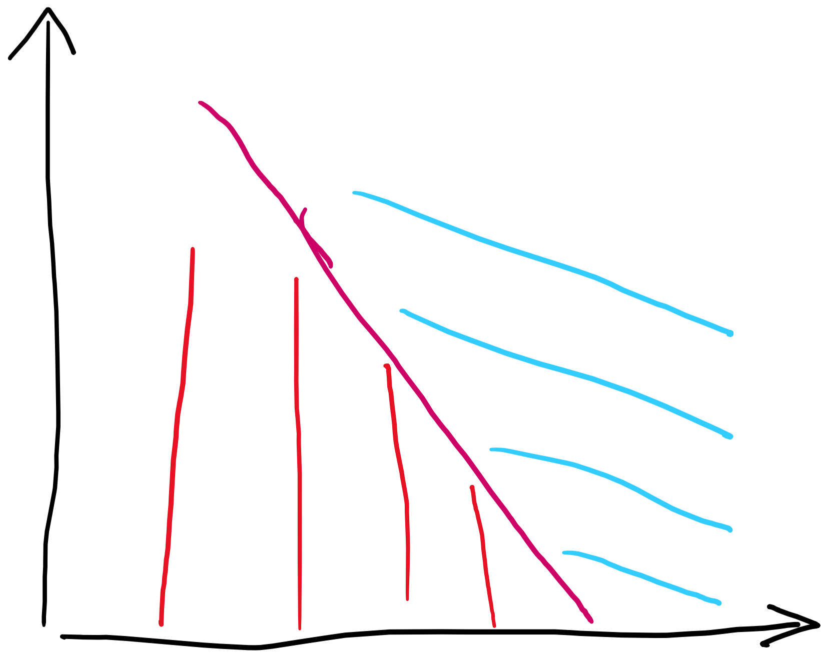
- Characteristics from the left must be at least as “fast” as the shock speed
- Characteristics from the right must at most as “slow” as the shock speed

Mathematically, this means that the **Lax entropy condition** holds

$$f'(u_l) \geq s \geq f'(u_r)$$

“Slower” means smaller speed s, “faster” means higher speed s





1. Example: Transport equation with constant speed:

We have $f'(u) = a$, so any discontinuity must move with speed a

2. Example: Burgers' equation

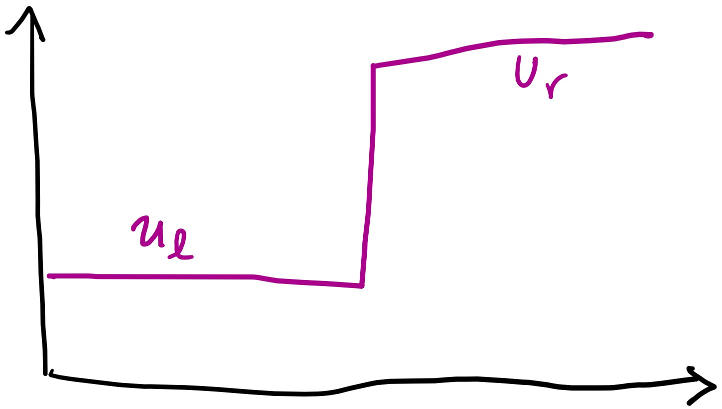
We have $f'(u) = 2u$, so we must have $u_l \geq u_r$

We can also use the RH condition.

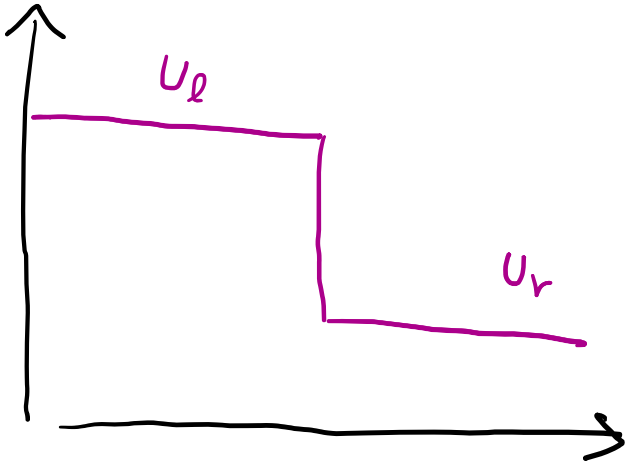
The characteristics associated with value u shall travel with speed s slower than the left shock but faster than the right shock:

$$\underbrace{\frac{f(u) - f(u_\ell)}{u - u_\ell}}_{s_\ell} \geq s \geq \underbrace{\frac{f(u) - f(u_r)}{u - u_r}}_{s_r}$$

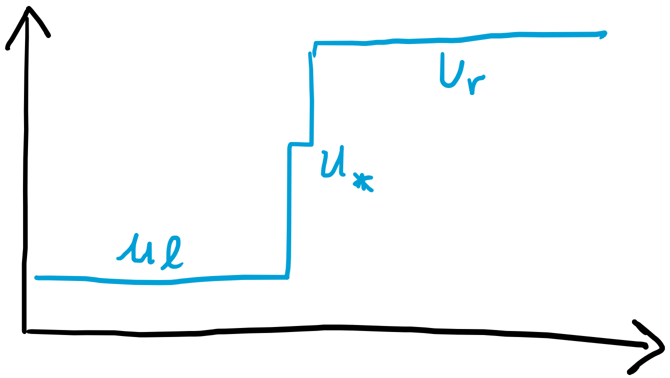
We can also identify "physical" solutions by the following argument, due to Liu. We start with discontinuous initial data:

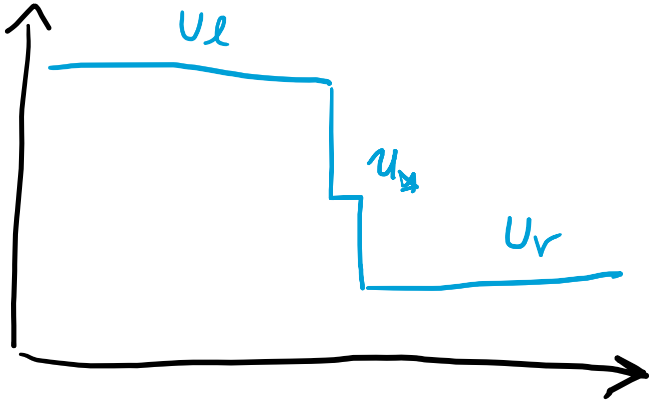


or



Suppose we perturb the initial data by an intermediate state:





Intuitively, the small perturbation at the start should not amplify over time.

Otherwise, we cannot hope for any reasonable modeling or simulation.

This requires that the jump behind travels not slower than the jump ahead.

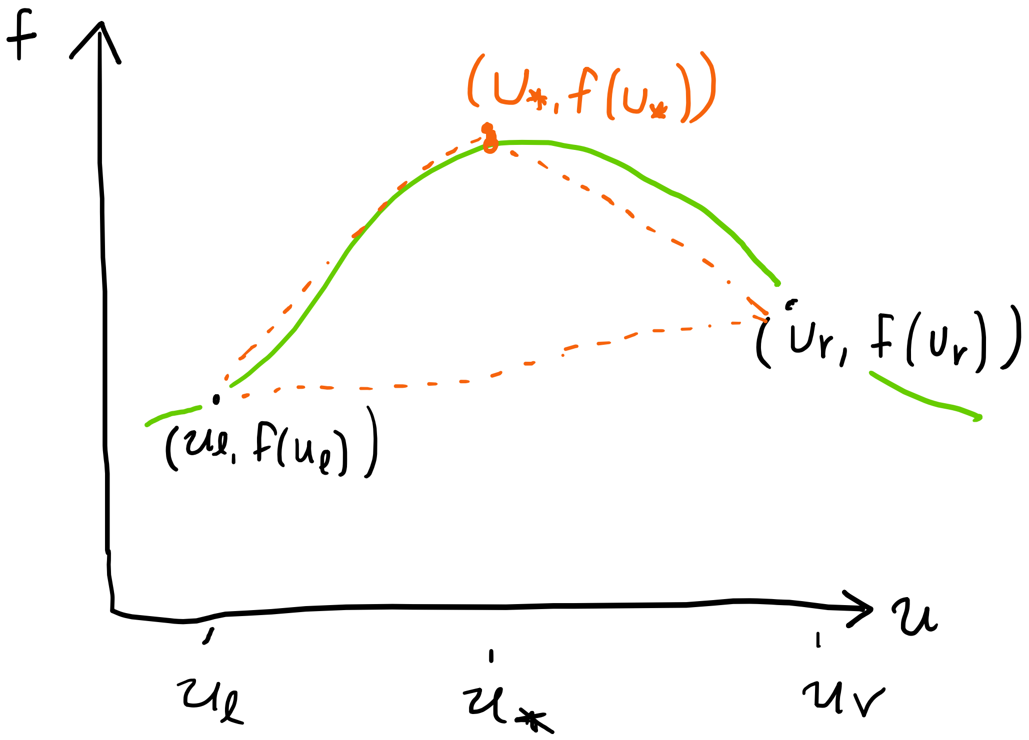
Using the Rankine-Hugoniot condition, this requirement reads:

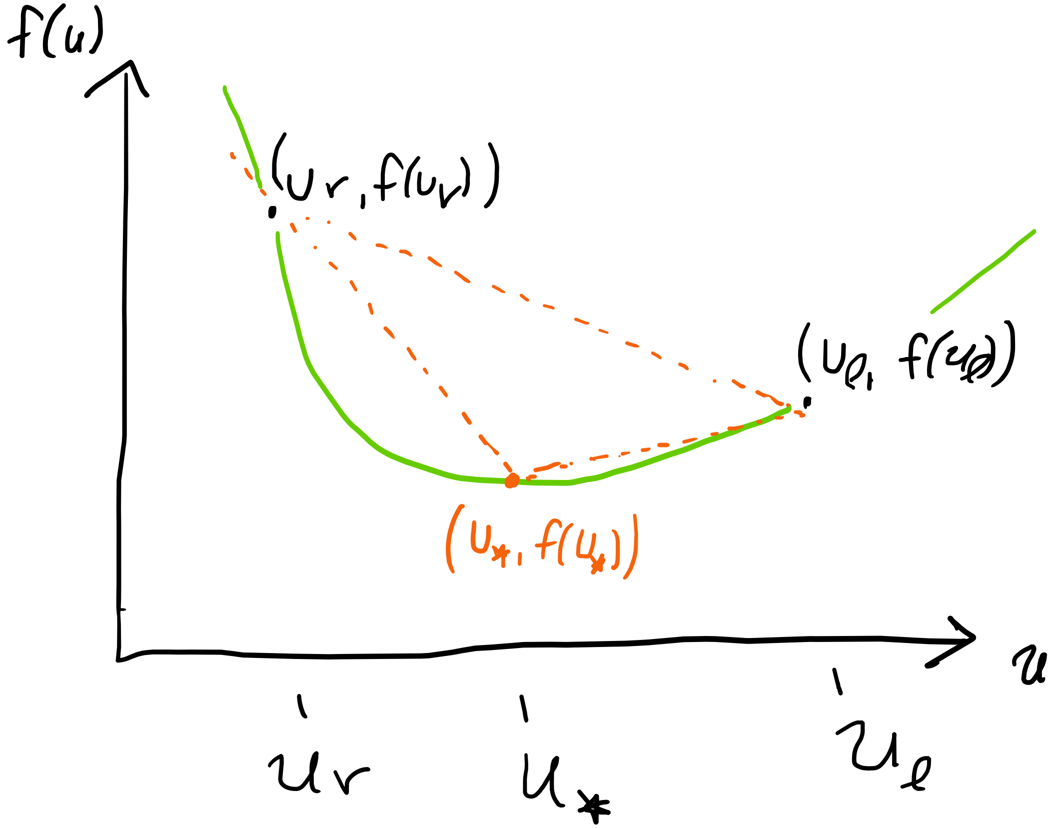
$$\frac{F(u^*) - f(u_\ell)}{u^* - u_\ell} \geq \frac{f(u_r) - f(u_*)}{u_r - u_*}$$

Liu's condition means that a shock is physically admissible if the states u_l and u_r satisfy the condition above, that is, the shock is stable under perturbations.

Requiring that the jump behind travels faster than the jump ahead, therefore means:

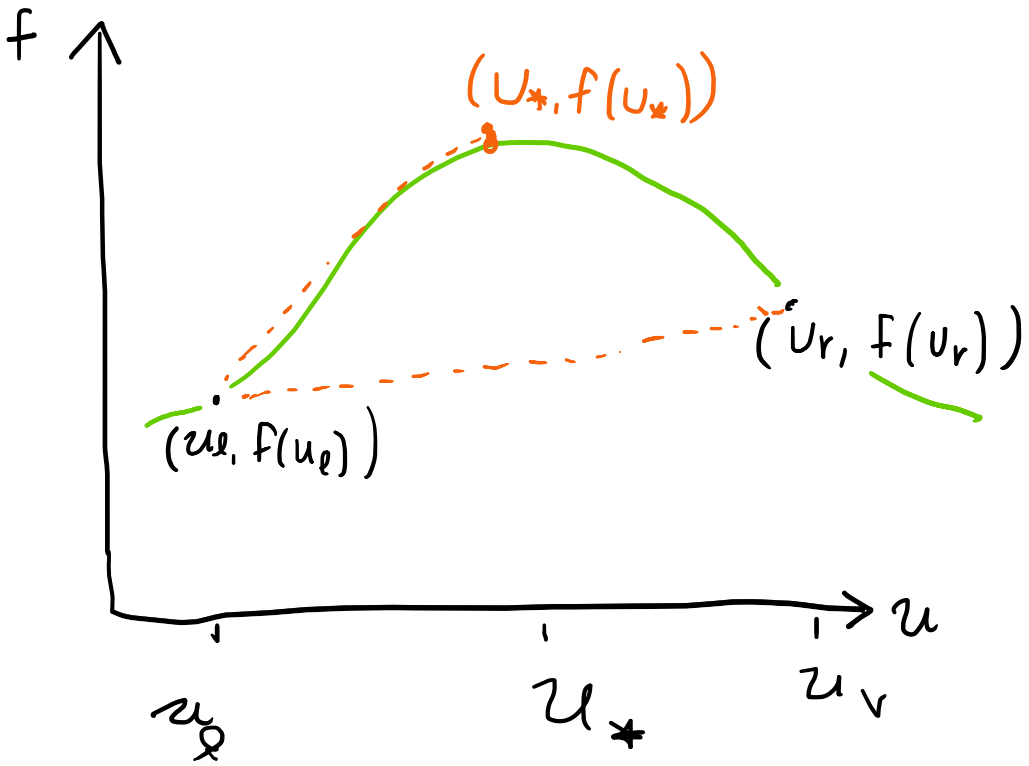
- If $u_l < u_r$, then f stays above or on the secant line
- If $u_l > u_r$, then f stays below or on the secant line

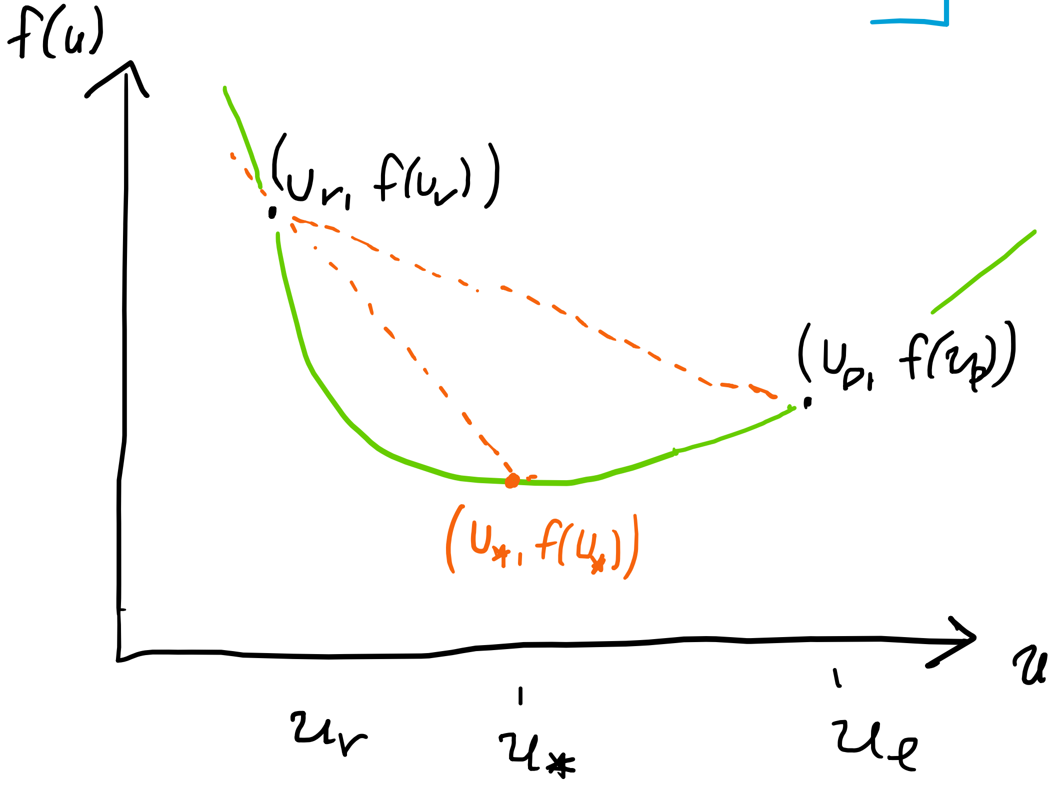




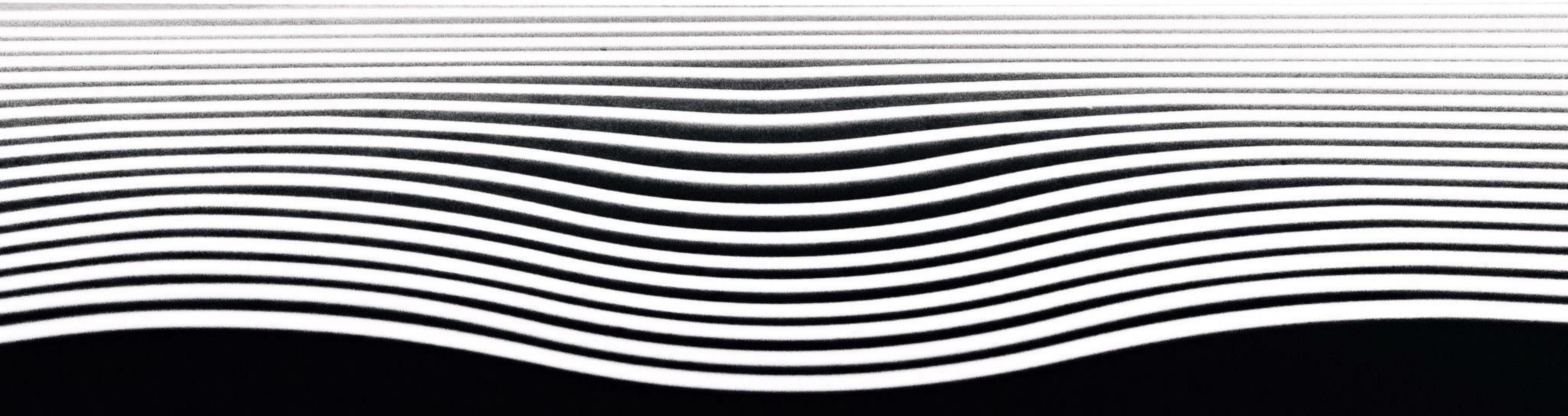
Equivalently, we can require that the speed of the original shock is not larger than the speed of the intermediate shock:

$$\frac{f(u^*) - f(u_\ell)}{u^* - u_\ell} \geq \frac{f(u_r) - f(u_\ell)}{u_r - u_\ell}$$





Numerical methods for conservation laws
4: Entropy conditions, Part 2



Towards Entropy solutions

The correct weak solutions are so-called entropy solutions. This theory is set up in several stages.

1. Entropy pairs
2. Weak entropy solutions
3. Vanishing viscosity limit
4. Contraction property
5. Existence
6. Additional properties

Entropy pairs

We say that two differentiable functions

$$\eta: \mathbb{R} \to \mathbb{R}, \qquad \Psi: \mathbb{R} \to \mathbb{R}$$

are an entropy-entropy flux pair (η, Ψ) compatible with the flux f if the entropy η is convex and

$$\eta'(u) \cdot f'(u) = \Psi'(u)$$

We do not need to assume that η, Ψ, or f are differentiable when this equation holds in the sense of distributions.

Note that entropy-entropy flux pairs always depend on the flux.

Examples

1) We take $\eta(u) = u$ and $\Psi = f$

2) Given any convex differentiable function η, we may pick

$$\Psi(\mathrm{u}) = \int_{z_0}^{u} \eta'(z)\, f'(z) dz$$

3) The Kruskov entropy-entropy flux pairs are

$$\eta(u) = |u - k|, \qquad \Psi(u) = \mathrm{sgn}(u - k)(\, f(u) - f(k)\,)$$

These are important but not differentiable. But convex functions can be approximated by linear combinations of such entropy functions

Entropy solutions

Suppose that $u \in L^1(\mathbb{R} \times \mathbb{R}_0^+)$ is a weak solution the conservation law with continuous flux f and integrable initial data $u_0 \in L^\infty(\mathbb{R})$.

We call u a weak entropy solution if for all entropy pairs (η, Ψ)

$$\int_0^\infty \int_{\mathbb{R}} \eta(u) \cdot \phi_t + \Psi(u) \cdot \phi_x \, dx \, dt + \int_{\mathbb{R}} \eta(u_0) \cdot \phi \, dx \ \geq 0$$

for all non-negative test functions ϕ.

Note: When a weak entropy solution u is a classical solution, then the entropy $\eta(u)$ is conserved:

$$
\begin{aligned}
0 &= \eta'(u)\left(\partial_t u + \partial_x f(u) \right) \\
&= \eta'(u)\, \partial_t u + \eta'(u)\, f'(u)\, \partial_x u \\
&= \partial_t \eta(u) + \Psi'(u)\, \partial_x u \\
&= \partial_t \eta(u) + \partial_x \Psi(u)
\end{aligned}
$$

(Now integration by parts...)

Note:

When u is weak entropy solution, then definition of weak entropy solutions implies that entropy is non-increasing:

$$\partial_t \eta(u) + \partial_x \Psi(u) \leq 0$$

(just another way of writing down the definition of weak entropy solutions, but in the sense of distributions)

This is why we call η the entropy.

Vanishing Viscosity limit

Suppose we have smooth solutions $u_\epsilon(x,t)$ to the viscous equation

$$\partial_t u_\epsilon + \partial_x f(u_\epsilon) = \epsilon \Delta u_\epsilon, \qquad\qquad u_\epsilon(x,0) = u_0(x).$$

Suppose that $u_\epsilon \in L^\infty$ is uniformly bounded for $0 \leq \epsilon \leq 1$.

If u_ϵ converges to $u \in L^\infty$ almost everywhere, then the limit u is a weak entropy solution.

Proof

Evans. PDE 11.4.2

First, suppose that η and ψ are smooth. Since

$$\partial_t u_\varepsilon + \partial_x f(u_\varepsilon) = \varepsilon \partial_{xx}^2 u_\varepsilon,$$

we can multiply by $\eta'(u)$ and get

$$\partial_t \eta(u_\varepsilon) + \partial_x \psi(u_\varepsilon) = \varepsilon\, \eta'(u_\varepsilon)\, \partial_{xx}^2 u_\varepsilon$$

Proof

We observe

$$\varepsilon\, \eta'(u_\varepsilon)\, \partial_{xx}^2 u_\varepsilon = \varepsilon\, \partial_{xx}^2 \Big(\eta(u_\varepsilon) \Big) - \varepsilon\, \eta''(u_\varepsilon)\, \partial_x u_\varepsilon\, \partial_x u_\varepsilon$$

Since η is convex, $\eta'' \geq 0$. Hence

$$\eta''(u_\varepsilon)\, \partial_x u_\varepsilon\, \partial_x u_\varepsilon \geq 0$$

Proof

We pick any non-negative test function φ with support in $\mathbb{R} \times (0,\infty)$.

We multiply the "entropy conservation law" by φ, integrate, and integrate by parts. Thus

$$\int_0^\infty \int_{\mathbb{R}} \eta(u_\varepsilon)\, \varphi_t + \psi(u_\varepsilon)\, \varphi_x \, dx dt$$

$$= \int_0^\infty \int_{\mathbb{R}} \varepsilon\, \eta''(u_\varepsilon)\, \partial_x u_\varepsilon\, \partial_x u_\varepsilon\, \varphi - \varepsilon\, \eta(u_\varepsilon)\, \varphi_{xx} \, dx dt$$

Proof

Hence

$$\int_0^\infty \int_{\mathbb{R}} \eta(u_\varepsilon)\, \varphi_t + \psi(u_\varepsilon)\, \varphi_x \, dx dt \geq -\int_0^\infty \int_{\mathbb{R}} \varepsilon\, \eta(u_\varepsilon)\, \varphi_{xx} \, dx dt$$

Recall that u_ε is uniformly bounded and η and ψ are smooth.
Letting $\varepsilon \to 0$ and using dominated convergence

$$\int_0^\infty \int_{\mathbb{R}} \eta(u)\, \varphi_t + \psi(u)\, \varphi_x \, dx dt \geq 0$$

Proof

Lastly, if η and ψ are not smooth, we can use smooth approximations and take the limit.

$\square$

This indicates the weak entropy solutions are the correct solutions, since many conservation laws arise via the vanishing viscosity simplification.

Establishing the almost everywhere convergence of u_ϵ towards u is the major challenge in using the vanishing viscosity approach.

Contraction property

Suppose that $u_0, v_0 \in L^\infty(\mathbb{R})$ and $u, v \in L^\infty(\mathbb{R} \times \mathbb{R}_0^+) \cap C(\mathbb{R}_0^+, L^1(\mathbb{R}))$ are the respective weak entropy solutions to the conservation laws

$$\partial_t u + \partial_x f(u) = 0, \qquad u(x,0) = u_0(x),$$

$$\partial_t v + \partial_x f(v) = 0, \qquad v(x,0) = v_0(x).$$

Then

$$\partial_t \|u(t) - v(t)\|_{L^1} \leq 0$$

This implies uniqueness of weak entropy solution.

Kruzkov, 1970.

Contraction property

Suppose that $u, v \in L^\infty(\mathbb{R} \times \mathbb{R}_0^+) \cap C(\mathbb{R}_0^+, L^1\ (\mathbb{R}))$ are two respective weak entropy solutions to the conservation law

$$\partial_t u + \partial_x\, f(u) = 0, \qquad\qquad u(x,0) = u_0(x),$$

We want to show $u = v$. Note that the initial conditions are the same. Recall that

$$\partial_t \| u(t) - v(t) \|_{L^1} \leq 0$$

While for $t = 0$, we have

$$\| u(0) - v(0) \|_{L^1} = \| u_0 - u_0 \|_{L^1} = 0$$

Since $\| u(t) - v(t) \|_{L^1}$ does not increase and is zero at $t = 0$, it must remain zero. In particular, $u(t) = v(t)$ at all times $t \geq 0$.

Entropy solutions: Existence

Suppose that $u_0 \in L^\infty(R)$ is a bounded measurable function, and that the flux is in C^2.

Then the scalar conservation law has a unique weak entropy solution u,

$$u \in L^\infty(\mathbb{R} \times \mathbb{R}_0^+) \cap C(\mathbb{R}_0^+, L_{loc}^1(\mathbb{R}))$$

Kruzkov, 1970.

Entropy solutions: Variation and Monotonicity

Suppose that $u_0 \in L^\infty(\mathbb{R})$ and that we have a weak entropy solution

$$u \in L^\infty(\mathbb{R} \times \mathbb{R}_0^+) \cap C(\mathbb{R}_0^+, L^1_{loc}(\mathbb{R}))$$

- If $u_0 \in BV(\mathbb{R})$, then $u(t) \in BV(\mathbb{R})$. What's $BV(\mathbb{R})$? Next slide
- If u_0 is monotonely decreasing/increasing, then so is u.

Conway & Smoller, 1966.

The total variation of a function measures how much of a function graph moves

- If $u : \mathbb{R} \to \mathbb{R}$ is differentiable, then

$$TV(u) = \int |\partial_x u(x)| \, dx$$

- If $u : \mathbb{R} \to \mathbb{R}$ is continuous, then

$$TV(u) = \sup \sum_i |f(x_{i+1}) - f(x_i)|$$

supremum over all partititions $\cdots \leq x_i \leq x_{i+1} \leq \cdots$ of $\mathbb{R}$

- If $u : \mathbb{R} \to \mathbb{R}$ is piecewise smooth with countably many jumps at $\dots \leq x_i \leq x_{i+1} \leq \dots$ then

$$TV(u) = \sum_i \int_{x_i}^{x_{i+1}} |\partial_x u| \, dx + \sum_i |\text{jump}(x_i)|$$

- Technically, we have

$$TV(u) = \sup \left\{ \int u(x) \, \varphi'(x) \, dx \;\middle|\; \begin{array}{l} \varphi \in C^1, \text{ compactly supported} \\ \|\varphi\|_\infty \leq 1 \end{array} \right\}$$

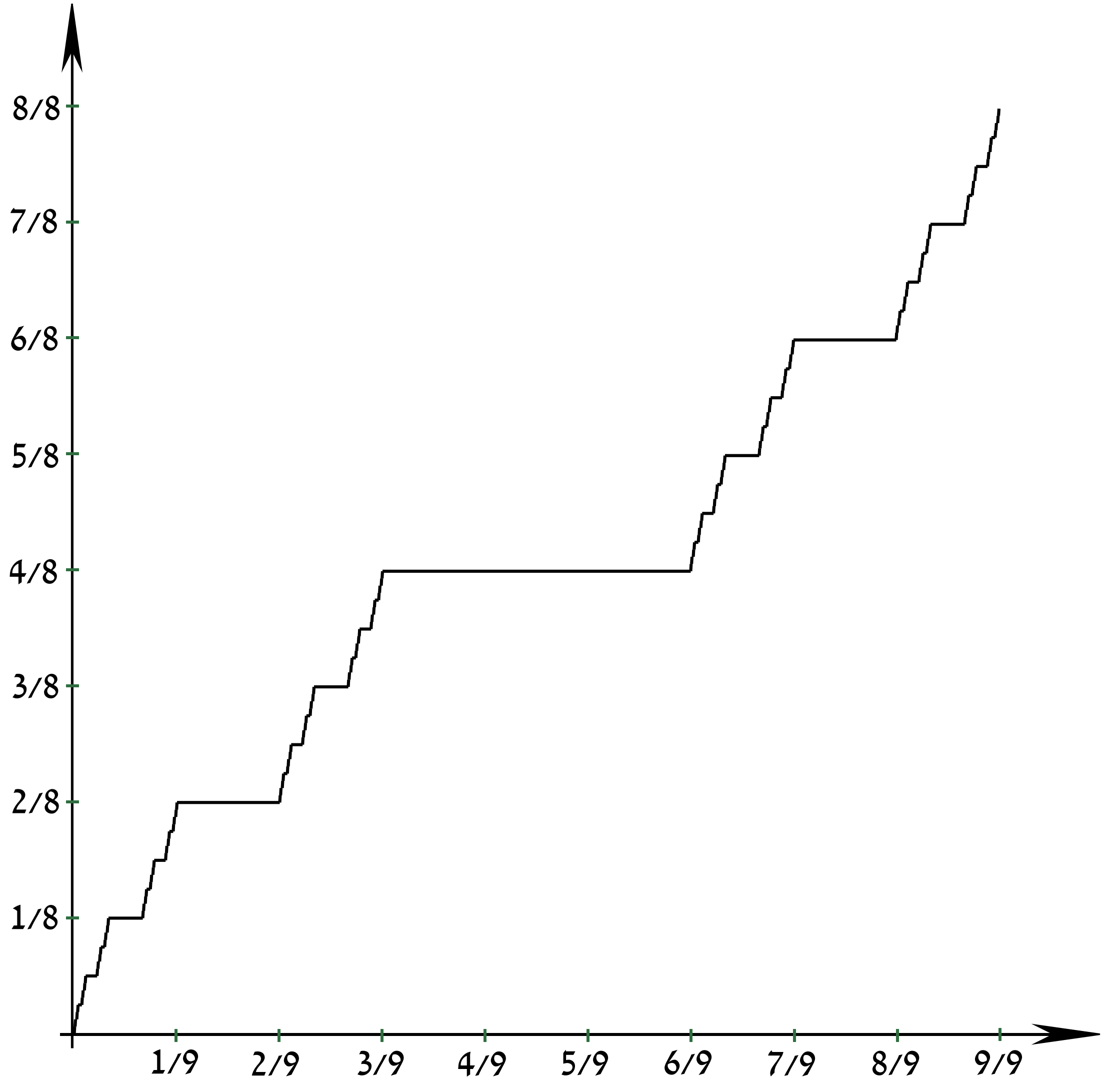
In most cases, u has enough structure so that a simpler definition can be used.

The space $BV(\mathbb{R})$ is the set of integrable functions u for which $TV(u) < \infty$.

We can equip that space with a norm

$$\|u\|_{BV} = \|u\|_{L^1} + TV(u)$$

$BV(\mathbb{R})$ is a Banach space.



Cantor function: a piecewise constant monotonely increasing function with countably many jumps and non-zero total variation.

Used for many counterexamples in real analysis

Main applications for conservation laws:

- The space $BV(\mathbb{R})$ is large enough to contain piecewise smooth functions.

- For piecewise smooth functions, possibly with jumps, we can easily compute the total variation.

- The variation of a solution does not blow up in finite time. For example, the sum of jumps does not explode.

Literature

O. Oleinik. Discontinuous solutions of non-linear differential equations

scalar in 1D

S. Kruzhkov. First-order quasilinear equations with several space variables

scalar in higher dimensions

S. Bianchini, A. Bressan. Vanishing viscosity solutions of nonlinear hyperbolic systems

systems in 1D (Annals of mathematics, 2005)

Hardcore analysis. Systems in higher dimensions seem to absent in the literature.

Literature

D. Serre, L. Silvestre. Multi-dimensional Burgers Equation with unbounded initial data: well-posedness and Dispersive Estimates

A. Bressan. Hyperbolic conservation laws: an illustrated Tutorial

L.C. Evans. Partial Differential Equations.

E. Tadmor. Approximate Solutions to nonlinear conservation laws and related equations.

”The basic questions regarding […] entropy solutions for general systems are still open. Instead, the present trend seems to concentrate on special systems with additional properties which enable to answer the questions of existence, stability, large time behavior, etc.”

Summary

- Classical solutions to restrictive for applications
- Weak solutions are good but too broad
- Entropy solutions are the weak solutions that are “physical”

- Definition via entropy-entropy flux pairs
- The limits of vanishing viscosity are entropy solutions
- L1 contraction property of entropy solutions
  - -> uniqueness!
- Existence
- Bounded variation