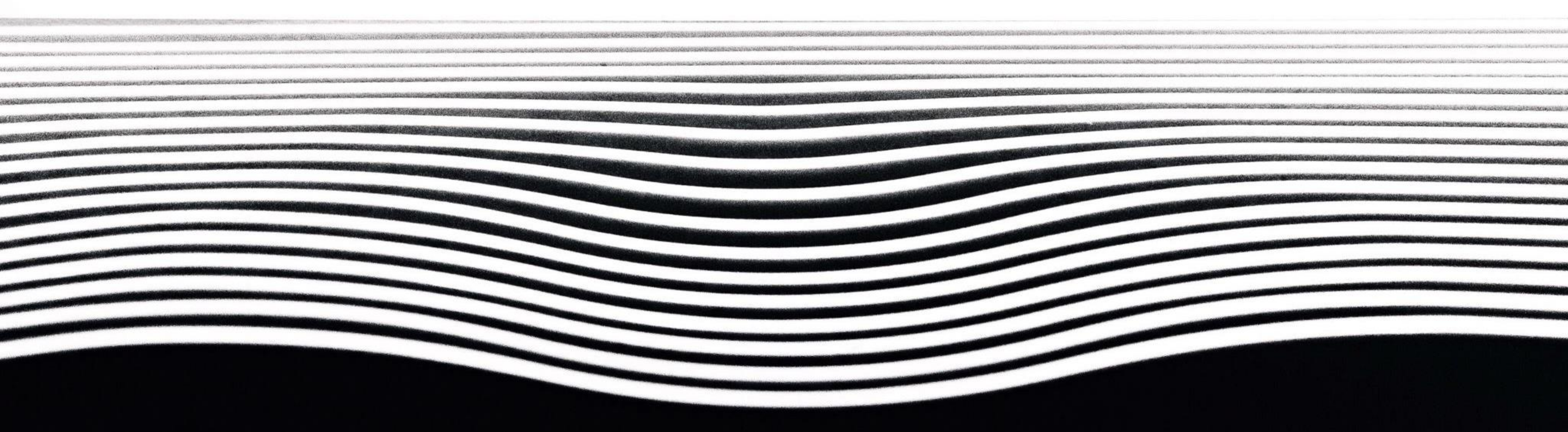
Numerical methods for conservation laws
3: Weak solutions and RH condition



Consider the conservation law

$$\partial_t u + \partial_x f = 0, \qquad\qquad u(x,0) = u_0(x)$$

We multiply that with a test function ϕ, that is, a smooth function with compact support in $R \times R_0^+$, and then we take integrals:

$$\int_0^\infty \int_{-\infty}^\infty \partial_t u(x,t) \cdot \phi(x,t) + \partial_x f(u,x,t) \cdot \phi(x,t)\, dxdt = 0$$

Both equations are equivalent for a smooth solution u. Now we integrate by parts

$$\int_0^\infty \int_{-\infty}^\infty u(x,t) \cdot \partial_t \phi(x,t) + f(u,x,t) \cdot \partial_x \phi(x,t)\, dxdt = -\int_{-\infty}^\infty u_0(x) \cdot \phi(x,0)\, dx$$

This is the weak formulation of the initial value problem. If any locally integrable function u satisfies the weak formulation for all test functions ϕ, then we call it a weak solution.

Recall a test function (in this context) is a function

$$\phi: R \times R_0^+ \to R$$

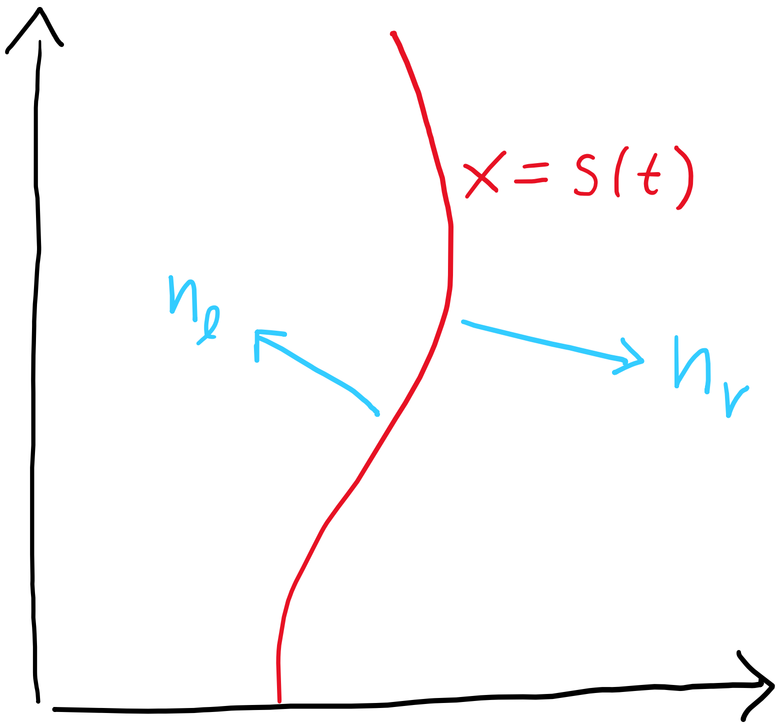
- such that ϕ has infinitely many derivatives in x and t, and
- the support of ϕ is compact. Here, the support is the set

$$supp(\phi) = \{\, (x,t) \in R \times R_0^+ : \ \phi(x,t) \neq 0 \,\}$$

Rankine-Hugoniot condition

We study the behavior of weak solutions that are (dis)continuous along (shock) curves.

Suppose that x = s(t) is a line parameterized in the time variable.



Speed: $\dot{s}(t)$

Slope: $\dfrac{1}{\dot{s}(t)}$

$$n_\ell = \begin{pmatrix} -1 \\ \dot{s}(t) \end{pmatrix}$$

$$n_r = \begin{pmatrix} 1 \\ -\dot{s}(t) \end{pmatrix}$$

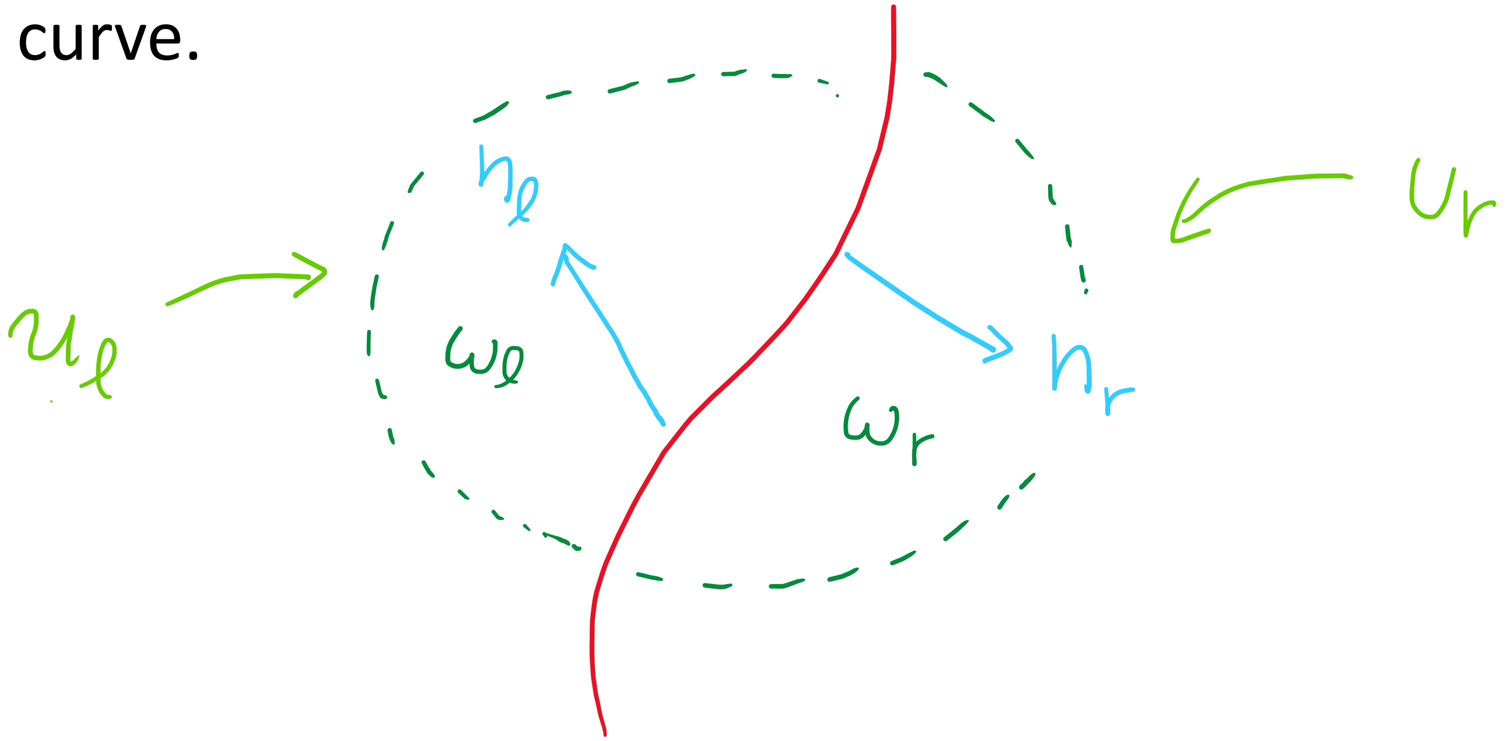
$\hat{n}_\ell$

$\hat{n}_r$

Consider a small domain ω that covers a part of the curve.

It splits into two pieces ω_l and ω_r, to the left and right of the curve, resp.

Suppose that $u(x,t)$ is a weak solution to a scalar conservation law with differentiable flux, taking differentiable values $u_l(x,t)$ and $u_r(x,t)$ to both sides of the curve.



For any test function ϕ supported within the small domain ω, we observe

$$0 = \iint_{\omega} u \varphi_t + f(u) \varphi_x \, dx dt$$

$$= \iint_{\omega_\ell} u \varphi_t + f(u) \varphi_x \, dx dt + \iint_{\omega_r} u \varphi_t + f(u) \varphi_x \, dx dt$$

$$= -\iint_{\omega_\ell} u_t \varphi + f(u)_x \varphi \, dx dt - \iint_{\omega_r} u_t \varphi + f(u)_x \varphi \, dx dt$$

$$+ \oint_{\partial \omega_\ell} \begin{pmatrix} f(u_\ell) \\ u_\ell \end{pmatrix} \hat{n}_r \cdot \varphi \, ds + \oint_{\partial \omega_r} \begin{pmatrix} f(u_r) \\ u_r \end{pmatrix} \hat{n}_\ell \cdot \varphi \, ds$$

Only the boundary integrals remain:

$$0 = \oiint_{\partial \omega_l} (-\dot{s} \cdot u_l + f(u_l))\, \phi\, ds + \oiint_{\partial \omega_r} (\dot{s} \cdot u_r - f(u_r))\, \phi\, ds$$

We conclude that this identity holds pointwise. We get

$$\dot{s} \cdot (u_l - u_r) = f(u_l) - f(u_r)$$

This is the **Rankine-Hugoniot** condition.

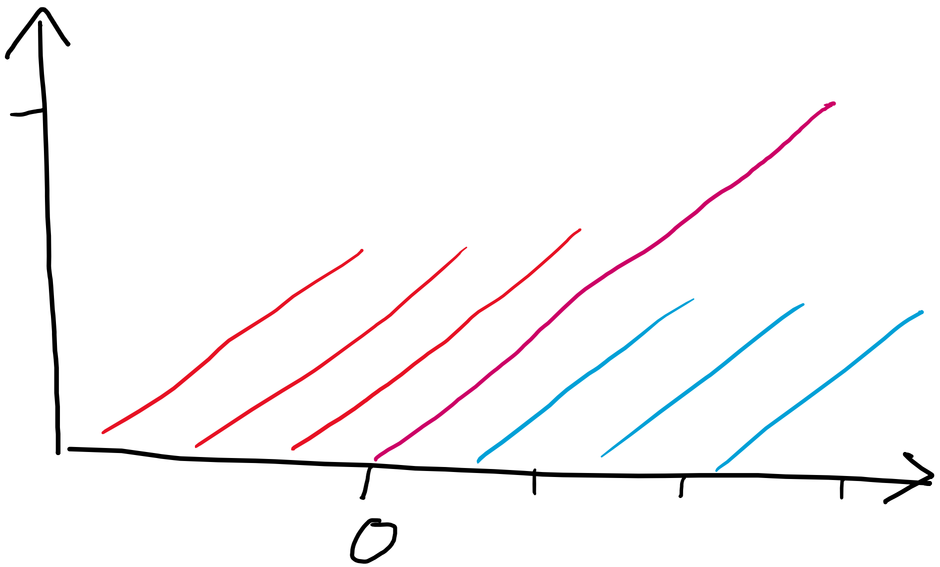
Main application: when $s(t)$ describes the trajectory of a shock curve, that is, a discontinuity of u.

Remember the transport equation

$$\partial_t u \; + a\partial_x u = \; 0$$

The initial values are piecewise constant with a discontinuity.

$$u_0(x) = \begin{cases} 2 & x < 0 \\ 1 & x > 0 \end{cases}$$



$$\dot{s} = \frac{a u_l - a u_r}{u_l - u_r} = a$$

Burgers' equation

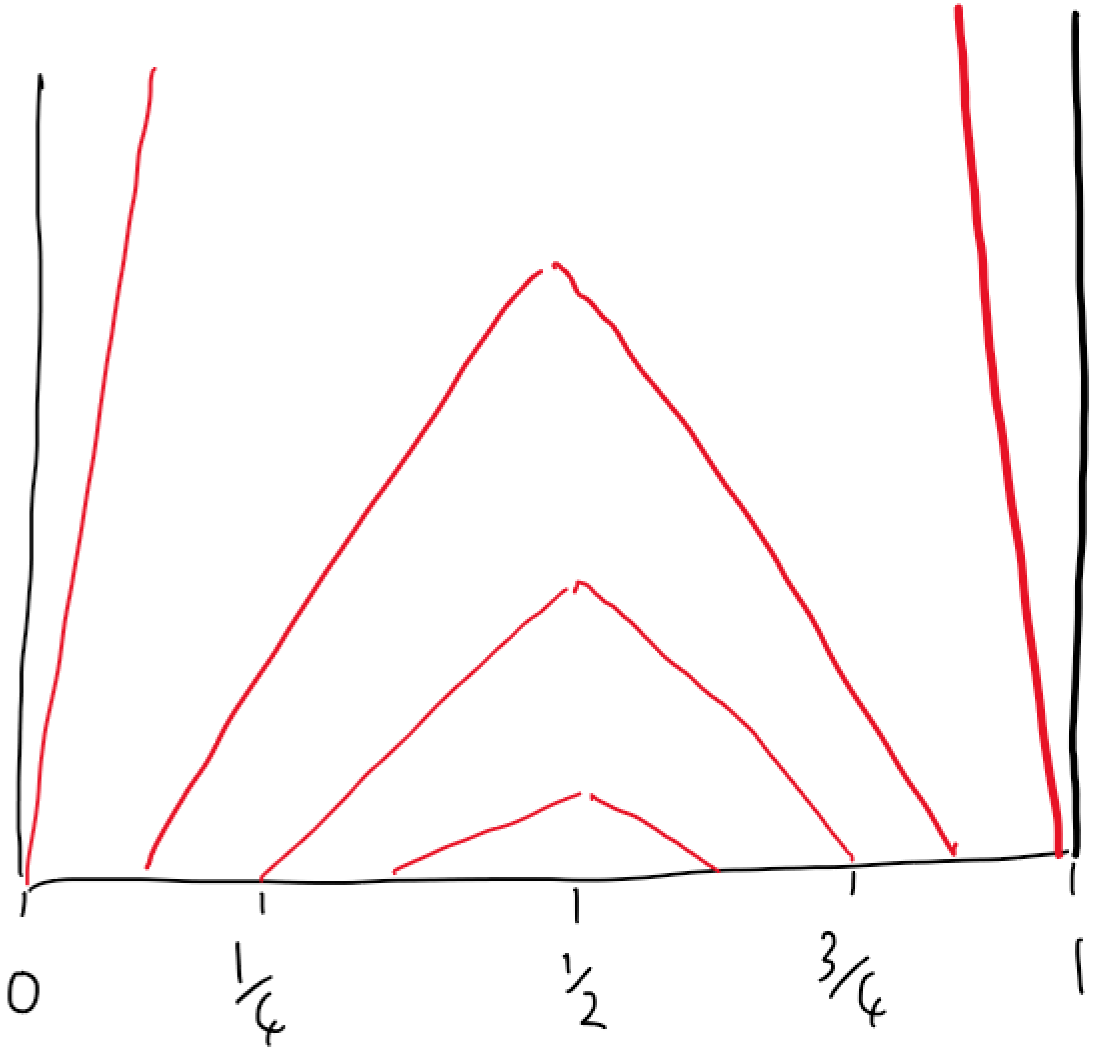
Example
Consider Burgers' equation with periodic BC over $[0,1]$ and with smooth initial data

$$u_0(x) = \sin(\, 2\,\pi x\,)$$

For the shock curve $x(t) = 0.5$ we get

$$\dot{s} \cdot (u_l - u_r) = f(u_l) - f(u_r) = 0$$

We conclude that the RH condition is satisfied.



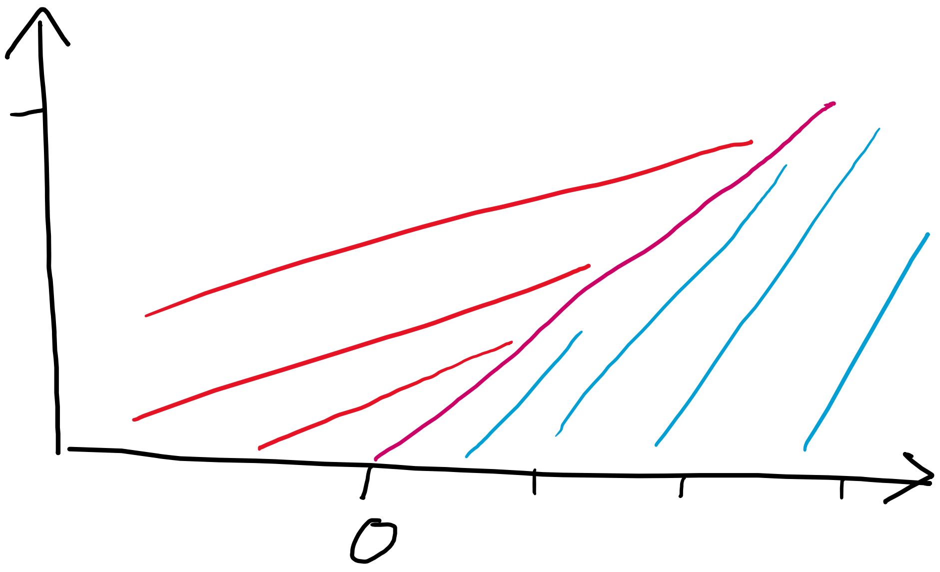
As an instructive example we consider, again, Burgers' equation.

$$\partial_t u + \partial_x(u^2) = 0$$

The initial values are piecewise constant with a discontinuity.

$$u_0(x) = \begin{cases} 2 & x < 0 \\ 1 & x > 0 \end{cases}$$

The characteristics crash into each other.



$$\dot{s}(2-1) = 2^2 - 1^2$$

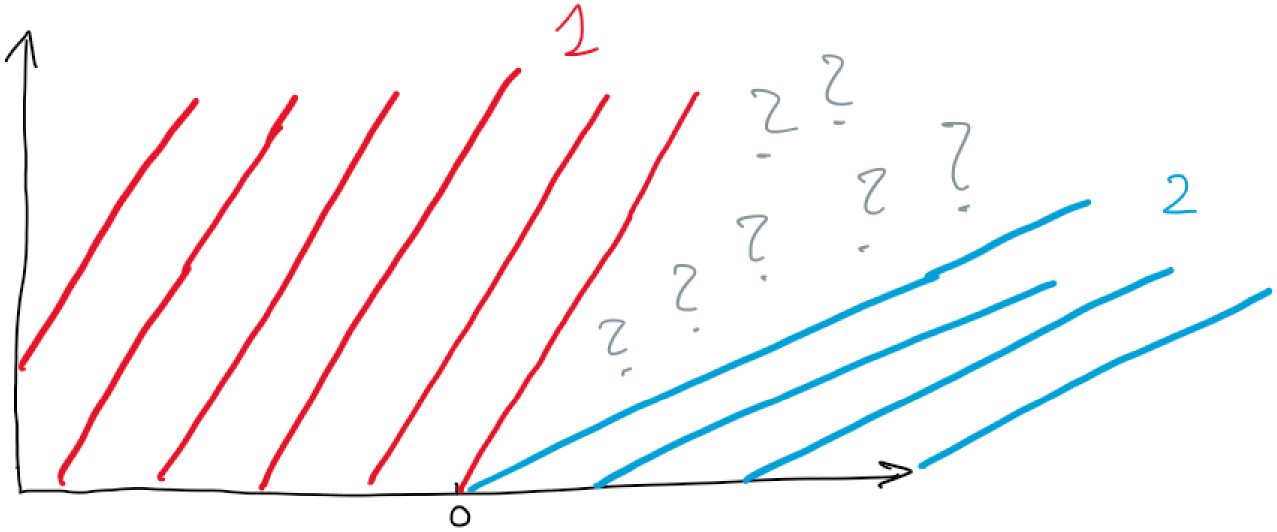
$$\Rightarrow \quad \dot{s} = 3$$

Once more, Burgers' equation

$$\partial_t u \; + \; \partial_{x(\, u^2)} = \; 0$$

with discontinuous piecewise constant initial values.

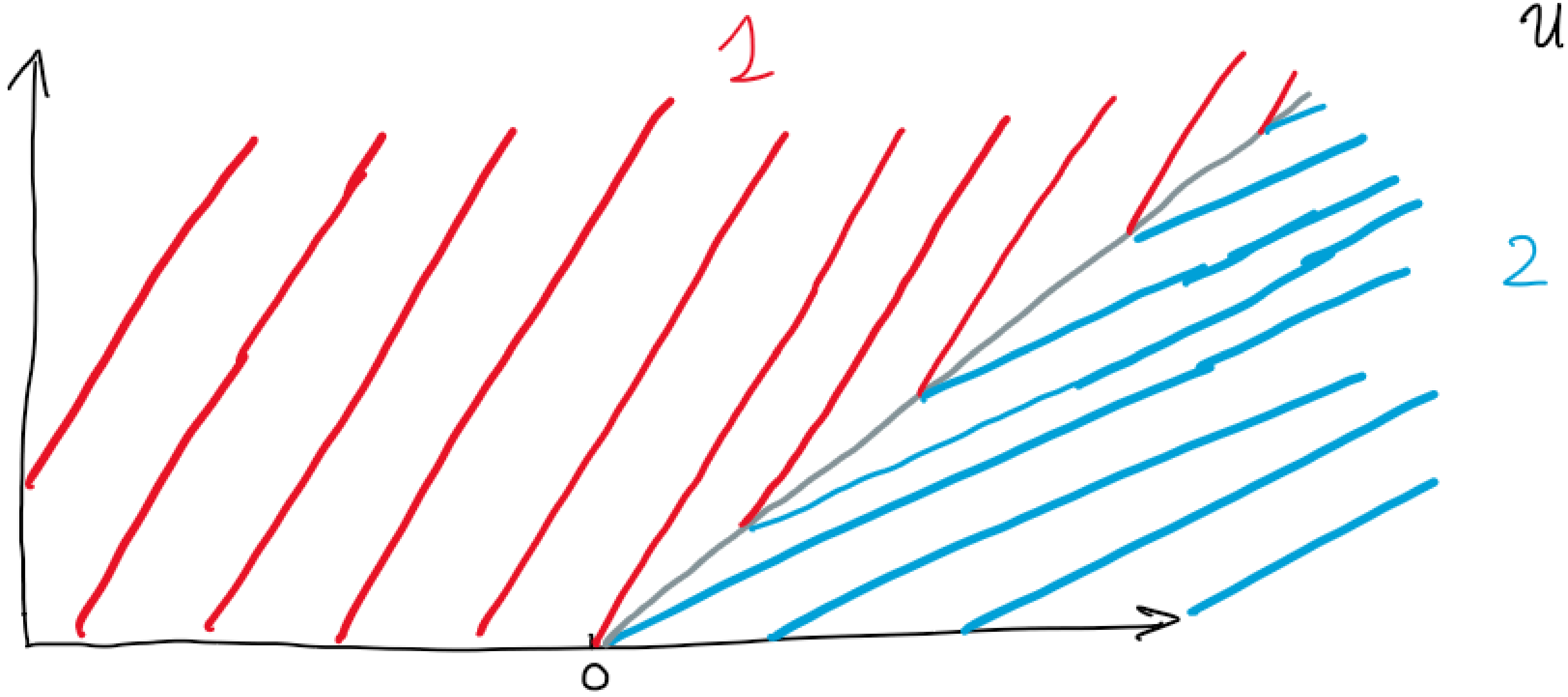
$$u_0(x) = \begin{cases} 1 & x < 0 \\ 2 & x > 0 \end{cases}$$



If there is a discontinuity along a straight line, its speed must be

$$\dot{s} = \frac{1-4}{1-2} = 3$$

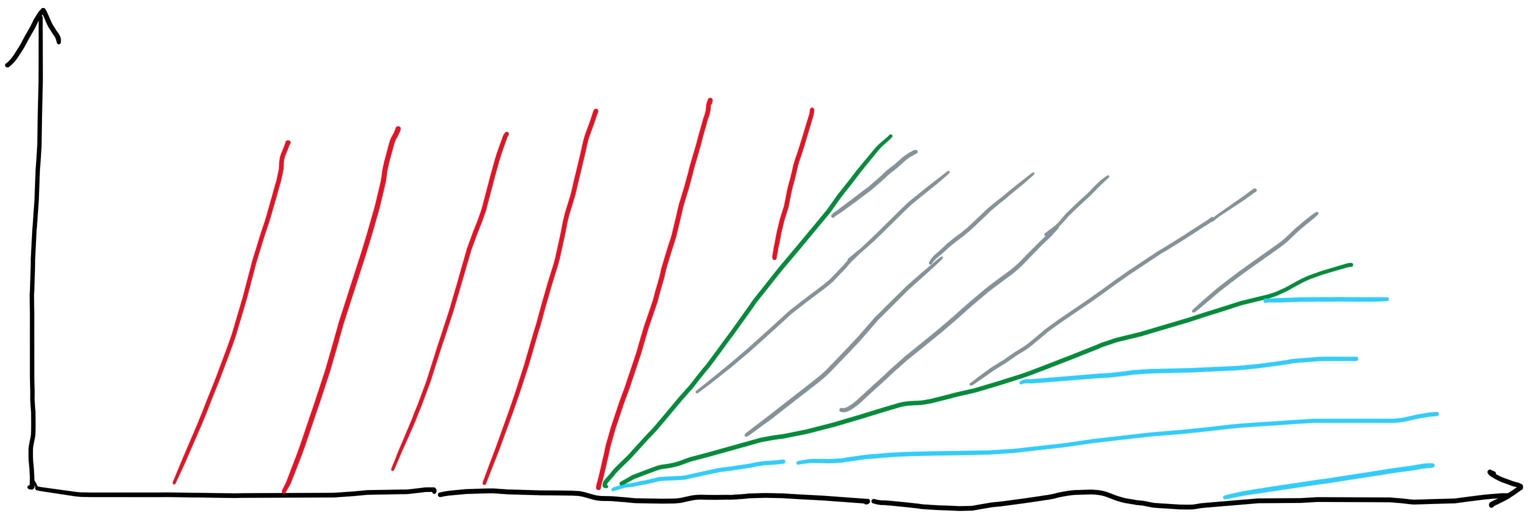
$$u(x,t) = \begin{cases} 1 & x \le \dot{s}t \\ 2 & x \ge \dot{s}t \end{cases}$$

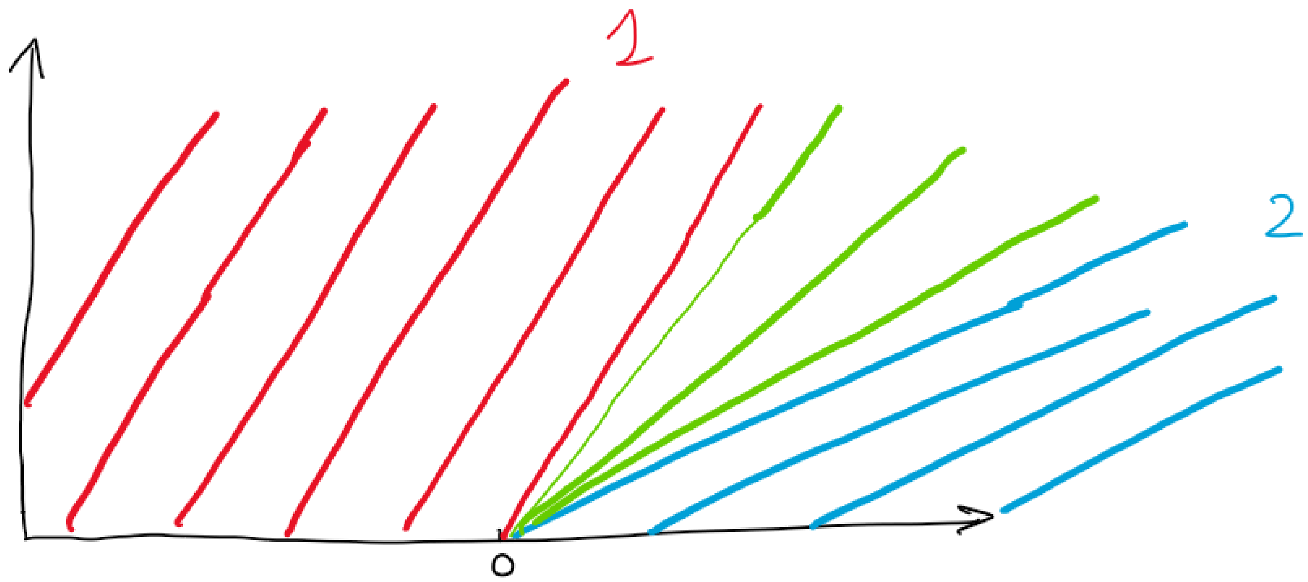


What about an intermediate value $u_m = 3/2 \in [1, 2]$?

We have $f'(u_m) = 3$ and two shock curves

$$s_l = \frac{f(u_m) - f(u_l)}{u_m - u_l} = \frac{5}{2} \qquad s_r = \frac{f(u_r) - f(u_m)}{u_r - u_m} = \frac{7}{2}$$





$$u(x,t) = \begin{cases} 1 & x \le 2t \\ x/2t & 2t \le x < 4t \\ 2 & x \ge 4t \end{cases} \underset{t>0}{=} \begin{cases} 1 & x/t \le 2 \\ \frac{1}{2} x/t & 2 \le x/t \le 4 \\ 2 & 4 < x/t \end{cases}$$

This is known as rarefaction wave.

Does it satisfy the conservation law?

1) u is constant when $x \le 2t$ or $4t \le x$

2) When $2t < x < 4t$, then $u(x,t) = \frac{x}{2t}$:

$$\partial_t u(x,t) + 2u(x,t) \cdot \partial_x u(x,t)$$

$$= -\frac{1}{2}\frac{x}{t^2} + 2 \cdot \frac{x}{2t} \cdot \frac{1}{2t} = -\frac{1}{2}\frac{x}{t^2} + \frac{1}{2}\frac{x}{t^2} = 0$$

3) The RH condition holds along $x = 2t$ and $x = 4t$

Trivial since u is continuous along those two lines

There are infinitely many weak solutions to this problem. That is typical for conservation laws.

So we have multiple weak solutions. How do we pick the one that's "physical"?

We want the characteristics to run into the shock, that is, no characteristics emerge from the shock wave.

Next time: Entropy conditions!